Angular Momentum

Angular momentum is the vector product of position and momentum:

$$\vec{L} \equiv \vec{r} \times \vec{p} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

The three cartesian components of angular momentum are:

$$L_x = y p_z - z p_y$$

$$L_y = z p_x - x p_z$$

$$L_z = x p_y - y p_x$$

The angular momentum may be possessed by

1. A rotating molecule
2. An electron orbiting around an atom
3. The spinning electrons and
4. Certain spinning nuclei

For a particle of mass m revolving around a point at a distance r, the angular momentum L is given by

$$L = mvr = mr^2\omega = I\omega$$

where, $v \rightarrow$ linear velocity

$\omega \rightarrow$ Angular "

$I \rightarrow$ Moment of inertia

K.E. of ~~the~~ the particle

$$E_k = \frac{1}{2} m v^2$$

$$= \frac{1}{2} m r^2 \omega^2$$

$$= \frac{1}{2} I \omega^2$$

$$= \frac{L^2}{2I}$$

In three dimensions, the angular momentum is represented by vector L.

$$L = r \times mv$$
$$= r \times p$$

$p \rightarrow$ linear momentum vector

$r \rightarrow$ vector from a fixed point to the mass point

L is perpendicular to the plane defined by r & p. From classical mechanics, it can be shown that the components of classical angular momentum vector are given by

$$L_x = y p_z - z p_y \quad \text{—} (1)$$

$$L_y = z p_x - x p_z \quad \text{—} (2)$$

$$L_z = x p_y - y p_x \quad \text{—} (3)$$

The square of the angular momentum

$$L^2 = L \cdot L = L_x^2 + L_y^2 + L_z^2$$

which is a scalar quantity.

If no torque is acting on the particle, its angular momentum remains constant in classical mechanics.

Classical mechanics permits all possible values of L. In quantum mechanics, the angular momentum is represented by an operator. The quantum mechanical operators for the components of angular momentum are obtained by replacing the quantities in equations ①, ② and ③ with their corresponding quantum mechanical operators.

Thus,

$$\hat{P}_x = -i\hbar \frac{\partial}{\partial x}, \quad \hat{P}_y = -i\hbar \frac{\partial}{\partial y}, \quad \hat{P}_z = -i\hbar \frac{\partial}{\partial z}$$

$$\therefore \hat{L}_x = -i\hbar \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \quad \text{—— (4)}$$

$$\hat{L}_y = -i\hbar \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \quad \text{—— (5)}$$

$$\hat{L}_z = -i\hbar \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \quad \text{—— (6)}$$

$$\therefore \hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \quad \text{—— (7)}$$

It is found ~~from~~ that according to Heisenberg's uncertainty principle, only square of the angular momentum, i.e,

L^2 and L_x

or L^2 and L_y

or L^2 and L_z can be simultaneously measured

Thus L^2 commutes with all of its components

$$\left[\hat{L}^2, L_x\right] = \left[\hat{L}^2, L_y\right] = \left[\hat{L}^2, \hat{L}_z\right] = 0 \quad (8)$$

On the other hand, $\hat{L}_x$ & $\hat{L}_y$
$\hat{L}_y$ & $\hat{L}_z$
$\hat{L}_z$ & $\hat{L}_x$ cannot be simultaneously measured. These operators do not commute with each other, i.e,

$$[\hat{L}_x, \hat{L}_y] \neq 0 \quad \text{— (9)}$$

$$[\hat{L}_y, \hat{L}_z] \neq 0 \quad \text{— (10)}$$

$$[\hat{L}_z, \hat{L}_x] \neq 0 \quad \text{— (11)}$$

If two operators commute with each other, ~~the~~ they have same eigenfunctions.

Thus, according to (8),
$\hat{L}^2$ & $\hat{L}_x$, $\hat{L}^2$ & $\hat{L}_y$, & $\hat{L}^2$ & $\hat{L}_z$ have same eigenfunctions.