

## Why do we study Graph Theory?

1. Branch of Discrete Mathematics with wide range of applications in practical world <sup>\*1</sup>
2. Defines the relationship between vertices and edges
3. Problems relating to discrete objects and their relationship can be converted into a Graph Theory problem.

## Graph [ Un-directed ]

Let  $V = \{v_1, v_2, \dots\}$  be a non-empty set of vertices and  $E = \{e_1, e_2, \dots\}$  be a set of edges.

Let  $\psi$  be a function that associates with each edge  $e_i$  an un-ordered pair of vertices  $(v_i, v_j)$  (not necessarily distinct).

Then the triplet  $(V, E, \psi)$  is called an **un-directed graph** <sup>\*2</sup> or graph  $G$ .

Let,  $e_k \in E$  &  $v_i, v_j \in V$  s. t.  $\psi(e_k) = \{v_i, v_j\}$ ; then  $e_k$  is associated with  $v_i, v_j$ .

Here  $v_i, v_j$  are adjacent vertices or end vertices for the edge  $e_k$ .

**Adjacent vertices:** Two vertices joined by the same edge.

**Adjacent edges:** Two edges incident on the same vertex.

**Self loop:** An edge having same vertex as its end vertices.

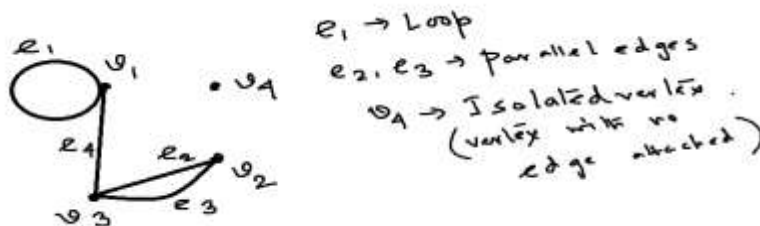
**Parallel edges:** More than one edges that are associated with a given pair of vertices.

If  $v_i$  is an end vertex for  $e_k$  then they are said to be incident with each other.

If  $V$  &  $E$  are both finite then the graph is a finite graph, otherwise it is infinite.

**Example 1:** Let  $V = \{v_1, v_2, v_3\}$  and  $E = \{e_1, e_2, e_3, e_4\}$ ,

With,  $\psi(e_1) = \{v_1, v_1\}$ ,  $\psi(e_2) = \{v_2, v_3\}$ ,  $\psi(e_3) = \{v_2, v_3\}$ ,  $\psi(e_4) = \{v_3, v_1\}$



<sup>\*1</sup> Problems in the fields of Physics, Chemistry and most importantly Computer Science can be re-modelled as graph theory problems.

<sup>\*2</sup>  $\psi$  is called the incidence function. If no ambiguity occurs an undirected graph is often referred to as  $G = (V, E)$ ; There is also a concept called the **Directed Graph**, to be discussed later.

**Order of a graph:** Number of vertices in a graph  $G$ , denoted by  $|V|$  or  $|G|$  or  $n$ .

**Size of a graph:** Number of edges in a graph  $G$ , denoted by  $|E|$  or  $\|G\|$  or  $m$ .

Order and size are finite if  $G$  and  $V$  are finite.

**Geometrical representation:**

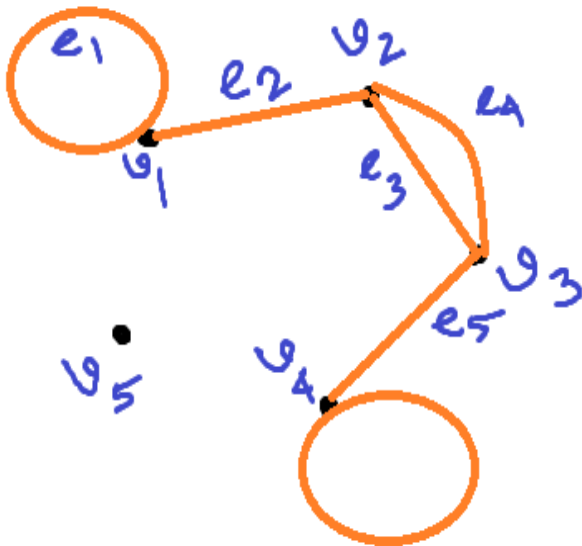
Each vertex is represented by a point and each edge by a line joining the adjacent vertices (which may or may not be straight lines).

Different layouts possible for the same graph.

Two or more edges may pass through a single point which may not represent a vertex.

**Example 2:** Let  $V = \{v_1, \dots, v_5\}$ ,  $E = \{e_1, \dots, e_6\}$

with  $\psi(e_1) = \{v_1, v_1\}$ ,  $\psi(e_2) = \{v_1, v_2\}$ ,  $\psi(e_3) = \{v_2, v_3\} = \psi(e_4)$ ,  $\psi(e_5) = \{v_3, v_4\}$ ,  $\psi(e_6) = \{v_4, v_4\}$



**Isolated vertex:** A vertex that is not associated with any edge.

**Null Graph:** A graph  $(V, E, \psi)$  is a null graph if  $E = \phi$ .

**Trivial Graph:** A graph with just one vertex and no edges.

Types of graph	Parallel edges	Loops
Pseudograph	Yes	yes
Multigraph	Yes	No
Simple graph	No	No

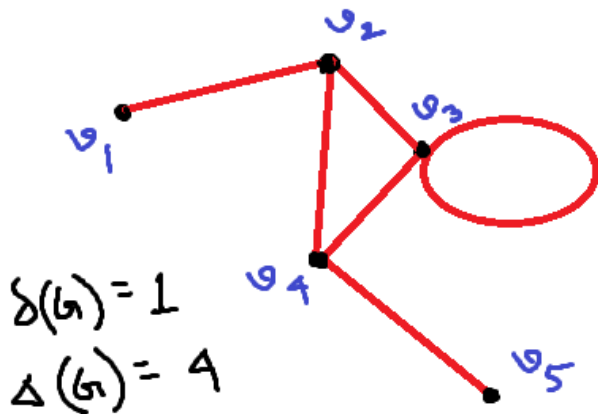
**Degree of a vertex:**

Number of edges incident on a given vertex  $v_i$ , with a loop assumed to add two to the degree count, denoted as  $d(v_i)$  or  $\text{deg}(v_i)$ . It is also known as **Valency**.

**Minimum degree of a graph  $G$ :**  $\delta(G) = \min\{d(v_i), \forall v_i \in V\}$

**Maximum degree of a graph  $G$ :**  $\Delta(G) = \max\{d(v_i), \forall v_i \in V\}$

**Example 3:** Minimum and maximum degree of a graph



A **Regular graph** is a graph in which all the vertices have the same degree.

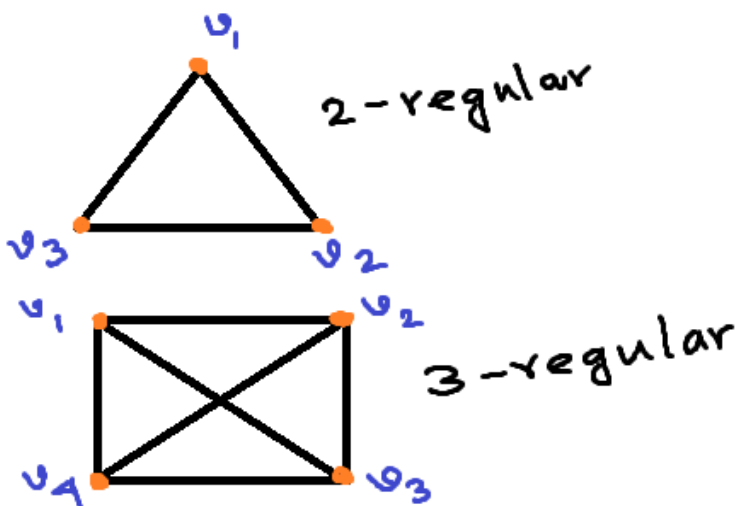
For a regular graph,  $\delta(G) = d(v_i) = \Delta(G), \forall v_i \in V$

A graph is  **$n$ -regular** if  $\delta(G) = \Delta(G) = n$

**Zero-regular graph** is a collection of isolated vertices.

**1-regular graph** is a collection of st. lines.

**Example 4:** 2-regular and 3-regular graphs



A vertex having degree 1 is a **pendant vertex** and an edge incident on it is called a **pendant edge**.

For a pendant vertex  $d(v_i) = 1$ .

A vertex  $v$  of a simple graph  $G$  is a **universal vertex** if  $d(v) = |V| - 1 = n - 1$

Universal vertex is also known as **dominating vertex**.

Collection of all dominating vertices in a simple graph  $G$  is called a **dominating set of vertices**.

A simple graph  $G$  is said to be a **complete graph** if every pair of distinct vertices in  $G$  is connected by an edge.

In a complete graph every vertex is a dominating vertex.

A complete graph of order  $n$  is denoted by  $K_n$ . A complete graph is also called a **Universal graph**.

A complete graph is also a regular graph. **Converse is however not true.**

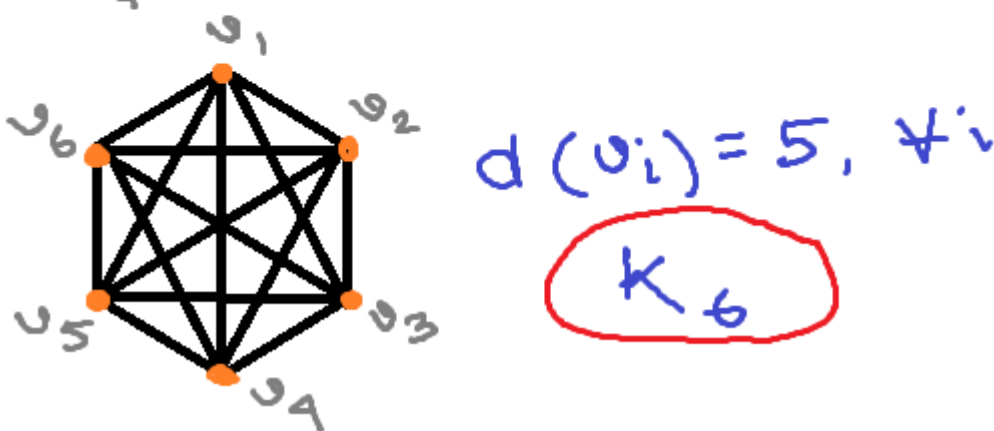
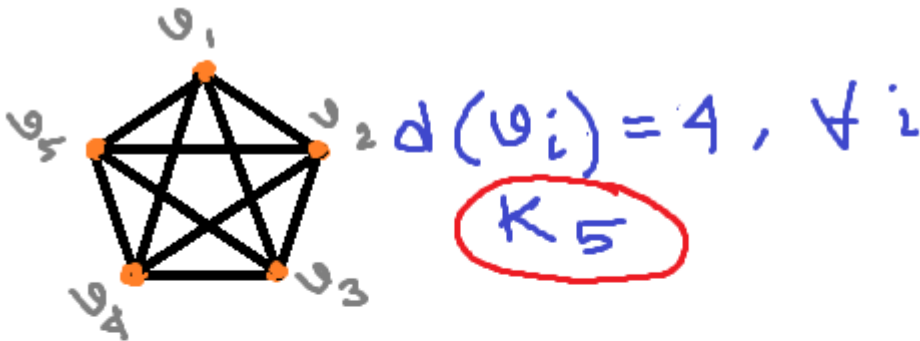
**A square is a regular graph with  $d(v_i) = 2, \forall i$ , but it is not a complete graph \*1.**

In Example 4 above both the graphs are complete graphs.

In the first one  $d(v_i) = 2 = n - 1 = 3 - 1$  [ $\because n = 3$ ], hence it is  $K_3$ .

In the second one  $d(v_i) = 3 = n - 1 = 4 - 1$  [ $\because n = 4$ ], hence it is  $K_4$ .

**Example 5: Complete graphs  $K_5$  &  $K_6$ .**



\*1 In a square  $d(v_i) = 2, \forall i$ , but to be a complete graph  $d(v_i)$  must be  $n - 1 = 4 - 1 = 3$  [in a square,  $n = 4$ ]

## Degree sequence of a graph:

Non-increasing (Non-decreasing) monotonic sequence of the degrees of the vertices.

A degree sequence is **graphic** if a simple graph can be constructed, using the given sequence.

**Result:** In any graph  $G$ ,  $\sum_{v \in V} d(v) = 2|E|$ .

**Case1:** If  $e$  is a loop then it adds 2 degrees to the overall degree count.

**Case2:** If not then  $e$  connects two vertices  $v_i, v_j$  and therefore adds 1 degree to  $v_i$  and 1 degree to  $v_j$  and therefore adds 2 degrees to the overall degree count.

In both cases therefore each edge contributes 2 degree to the overall degree count.

Hence the result. **[Handshaking Lemma]**

**Corollary:** If  $G$  is a  $p$ -regular graph with  $n$  vertices then  $|E| = \frac{np}{2}$

Since each of  $n$  vertices have the same degree  $p$ , hence  $\sum d(v) = np$ ; hence the result follows.

**Corollary:** for a degree sequence  $\{d_1, d_2, \dots, d_n\}$ ,  $\sum_{i=1}^n d_i$  must be even \*1.

Obvious! because otherwise  $|E|$  will be a fraction, which is not possible.

A vertex  $v_i$  is an **Odd** vertex if  $d(v_i)$  is odd and **Even** if  $d(v_i)$  is even.

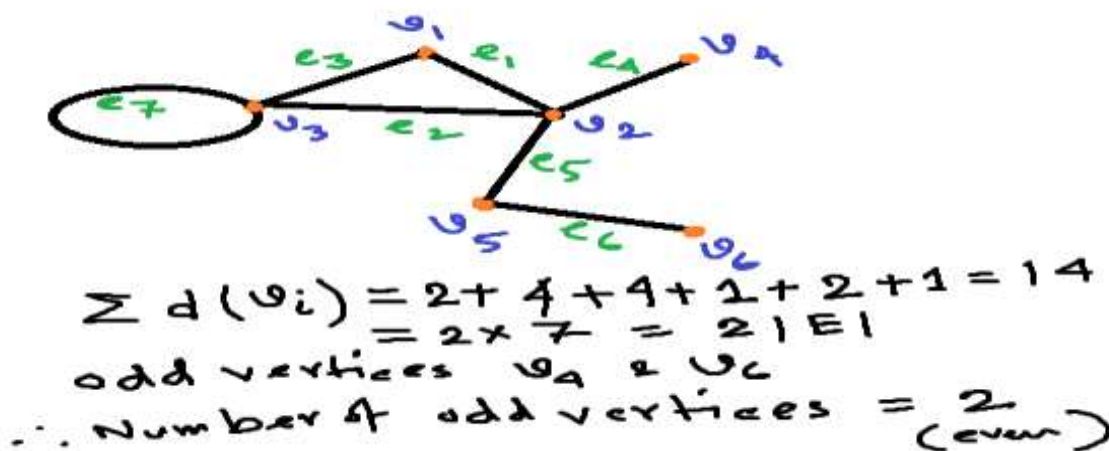
**Result:** In any graph  $G$ , the number of odd vertices must be even.

Let,  $V_1 = \{\text{set of odd vertices}\}, V_2 = \{\text{set of even vertices}\}$

Then,  $\sum d(v) = \sum_{V_1} d(v) + \sum_{V_2} d(v) \Rightarrow \sum_{V_1} d(v) = \sum d(v) - \sum_{V_2} d(v)$

Then,  $\sum_{V_1} d(v)$  must be even.

Again sum of odd number of odd numbers is always odd and sum of even number of odd numbers is even, then it follows that the number of terms in  $V_1$  must be even.



\*1 Converse is not true; i.e. if (sum of degrees) is even it may not be a valid degree sequence.

**Result:** In a simple graph  $G$ , with  $|V| = n$ ,  $d(v) \leq n - 1, \forall v \in V$ .

Since each vertex can be connected to at most  $n - 1$  number of vertices (as no self-loop & multiple edges are allowed), the result is obvious.

**Result:** Every simple graph with  $|V| = n \geq 2$ , must have at least a pair of vertices with same degree.

If possible let  $d(v_i) \neq d(v_j), i \neq j$  for each pair;

then the options for  $d(v)$  are  $0, 1, \dots, n-1$ ; i.e. a total of  $n$  options for  $n$  vertices; since no degree is repeated then there is a one-to-one correspondence between  $d(v_1), d(v_2), \dots, d(v_n)$  &  $0, 1, \dots, n - 1$ ;

then let  $d(v_r) = 0$  &  $d(v_s) = n - 1$ ;

which is impossible since  $d(v_s) = n - 1$  means that  $v_s$  is connected to all other vertices including  $v_r$  but  $d(v_r) = 0$  means  $v_r$  is an isolated vertex not connected with any other vertex;

hence the result follows.

**Corollary:** This result also proves that the degree sequence of a simple graph must have a pair of equal terms.

**Converse however is not true: "A pair of equal terms in the degree sequence does not guarantee a simple graph" There exists no simple graph with degree sequence (1,1,3,3)! [Do yourself]**

**Result:** For any graph  $G$ ,  $\delta(G) \cdot |V| \leq 2|E| \leq \Delta(G) \cdot |V|$

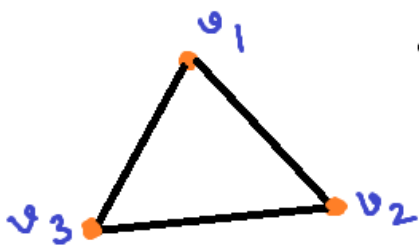
**Hint:**  $\sum d(v_i) = 2|E|$ ;  $\delta(G) \leq d(v_i) \leq \Delta(G), \forall v_i \in V$ ; hence,  $|V|\delta(G) \leq \sum d(v_i) \leq |V|\Delta(G)$

A simple graph consisting of  $n(\geq 3)$  vertices  $v_1, v_2, \dots, v_n$  and edges  $(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1)$  is a **Cycle** graph, denoted by  $C_n$ .

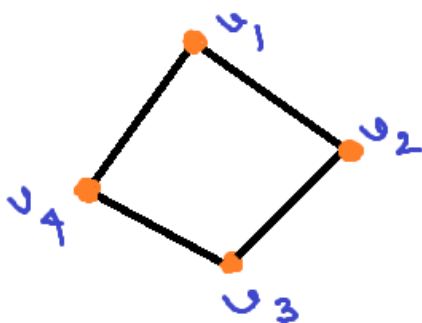
In a cycle  $C_n, d(v) = 2, \forall v \in V$ ; also, in a cycle  $C_n, |V| = |E| = n$ .

A cycle is even or odd according as  $n$  is even or odd.

A cycle graph with 1 or 2 vertices is not possible because 1 leads to a loop and 2 leads to parallel edges.



odd cycle  $C_3$   
 $d(v_i) = 2 \quad \forall i$   
 $|V| = |E| = 3$



even cycle  $C_4$   
 $d(v_i) = 2 \quad \forall i$   
 $|V| = |E| = 4$

### Subgraph:

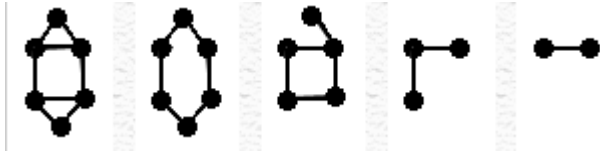
Let  $G(V, E)$  be a graph and let  $H = (V_1, E_1)$  be another graph. Then  $H$  is said to be a subgraph of  $G$  if,  $V_1 \subseteq V$  &  $E_1 \subseteq E$ .

A null graph is a **Trivial** subgraph of any graph  $G$ .

The graph  $G$  itself is an **Improper** subgraph of itself.

Any other subgraph will be called a **Proper** subgraph.

If  $H$  is a subgraph of  $G$  then  $G$  is called a **Supergraph** of  $H$ .

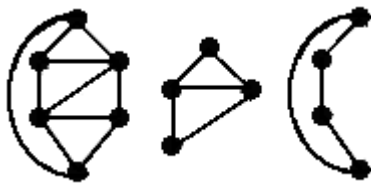


In the picture on the left all the graphs from 2<sup>nd</sup> to 5<sup>th</sup> are subgraphs of 1st

### Vertex induced subgraph:

Let  $G(V, E)$  be a graph and let  $V_1 \subseteq V$

Induced subgraph of  $G$  on  $V_1$  is a graph with vertex set  $V_1$  and edge set that connects the vertices of  $V_1$  in  $G$ , denoted by  $G(V_1)$ .

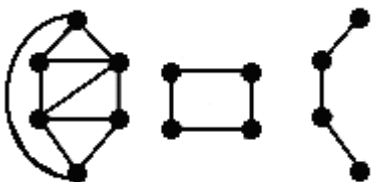


In the picture on the left 2<sup>nd</sup> and 3<sup>rd</sup> graphs are vertex-induced subgraphs of the 1st

### Edge induced subgraph:

Let  $G(V, E)$  be a graph and let  $E_1 \subseteq E$

Induced subgraph of  $G$  on  $E_1$  denoted by  $G[E_1]$  is a graph with edge set  $E_1$  and vertices of  $G$  that are incident with at least one edge of  $E_1$ .



In the picture on the left 2<sup>nd</sup> and 3<sup>rd</sup> graphs are edge-induced subgraphs of the 1st

### Edge deletion:

Let  $G(V, E)$  be a graph and let  $F \subset E$

The subgraph of  $G$  obtained by deleting  $F$  from  $E$ , keeping intact all their end vertices is a subgraph, denoted by  $G - F$  or  $G \setminus F$

i.e.  $G - F = (V, E - F)$

In particular if  $F = \{e\}$  then it is written as  $G - e$  or  $G \setminus e$ .

### Vertex deletion:

Let  $G(V, E)$  be a graph and let  $W \subset V$

The subgraph of  $G$  obtained by deleting  $W$  from  $V$ , along with all the edges incident on them, is a subgraph, denoted by  $G - W$  or  $G \setminus W$

i.e.  $G - W$  is a vertex induced subgraph induced by  $V - W$ .

In particular if  $W = \{v\}$ , then it is denoted by  $G - v$  or  $G \setminus v$ .



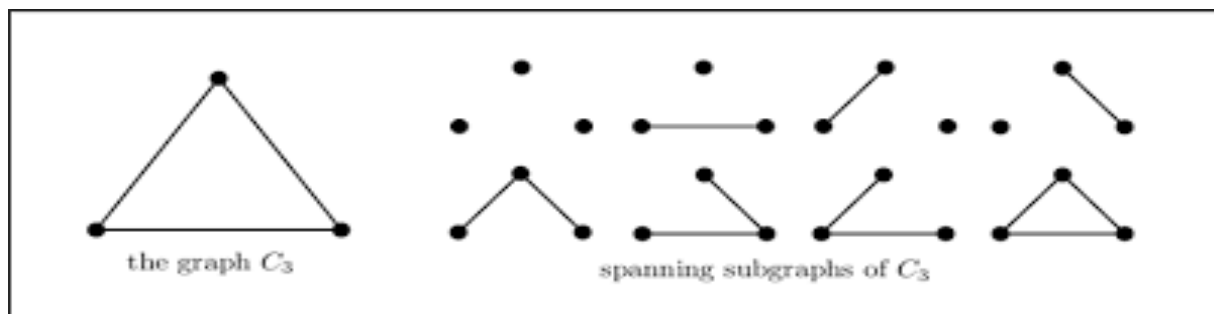
In the picture on the left the 2<sup>nd</sup> graph is a vertex-deleted subgraph of the 1<sup>st</sup>

And the 3<sup>rd</sup> graph is an edge-deleted subgraph of the 1<sup>st</sup>

### Spanning subgraph:

Let  $G(V, E)$  be a graph. A **spanning subgraph** is a subgraph that contains all the vertices of the original graph. Thus, a spanning subgraph is only possible by edge deletion. Let  $S$  be the set of deleted edges of  $G$ . Then the spanning subgraph of  $G$  is denoted as  $G \setminus S$ .

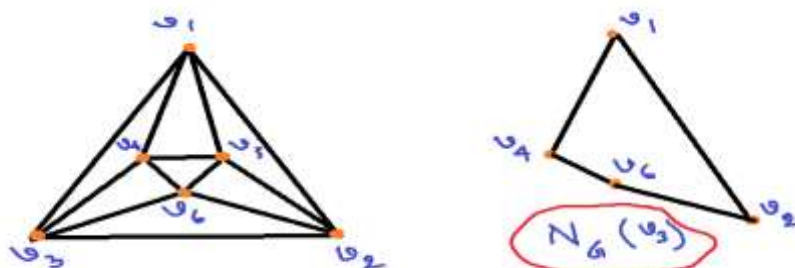
Every spanning subgraph is called a factor of  $G$ .



### Neighbourhood of a vertex:

Let  $G(V, E)$  be a graph and  $v \in V$ . Then **neighbourhood of the vertex**  $v$  is a vertex-induced subgraph of  $G$  induced by the set of vertices that are adjacent to  $v$ , denoted by  $N_G(v)$ .

In general,  $N_G(v)$  does not include  $v$  itself and is therefore assumed to be **Open**.



GRAPH THEORY - Preliminaries

Let  $G_1(V_1, E_1), G_2(V_2, E_2)$  be two graphs.

Then,  $G_1 \cup G_2 = \{V_1 \cup V_2, E_1 \cup E_2\}$  is a supergraph of  $G_1$  &  $G_2$ .

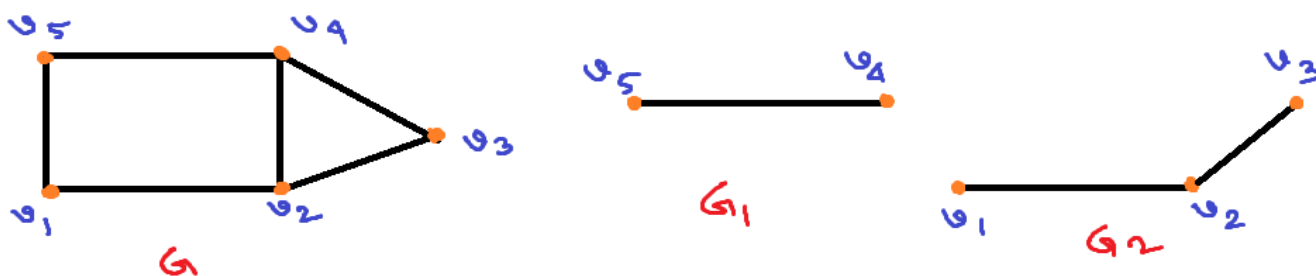
Also,  $G_1 \cap G_2 = \{V_1 \cap V_2, E_1 \cap E_2\}$  is a subgraph of  $G_1$  &  $G_2$ .

$G_1 \cap G_2$  is the largest common subgraph of  $G_1, G_2$ .

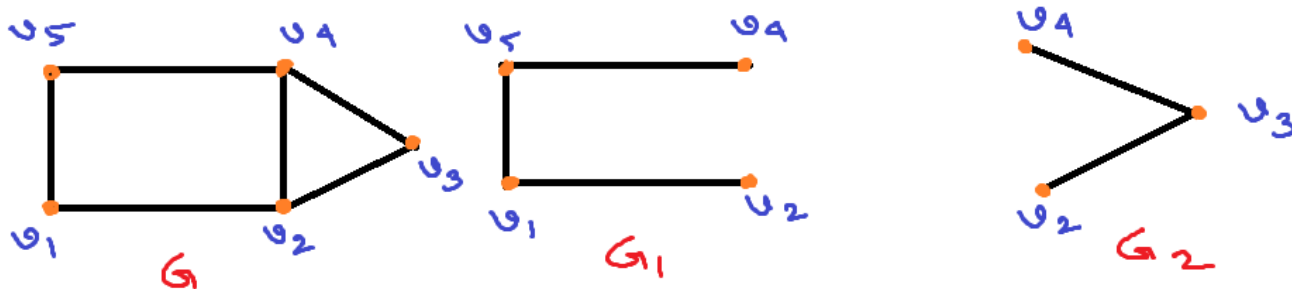
Also  $G_1 \cup G_2$  is the smallest supergraph containing  $G_1, G_2$ .

Let  $G(V, E)$  be a graph and  $G_1(V_1, E_1), G_2(V_2, E_2)$  are two subgraphs.

They are **edge disjoint** if  $E_1 \cap E_2 = \phi$  & **vertex disjoint** if  $V_1 \cap V_2 = \phi$ .



In the figure above  $G_1$  and  $G_2$  are vertex-disjoint subgraphs of  $G$

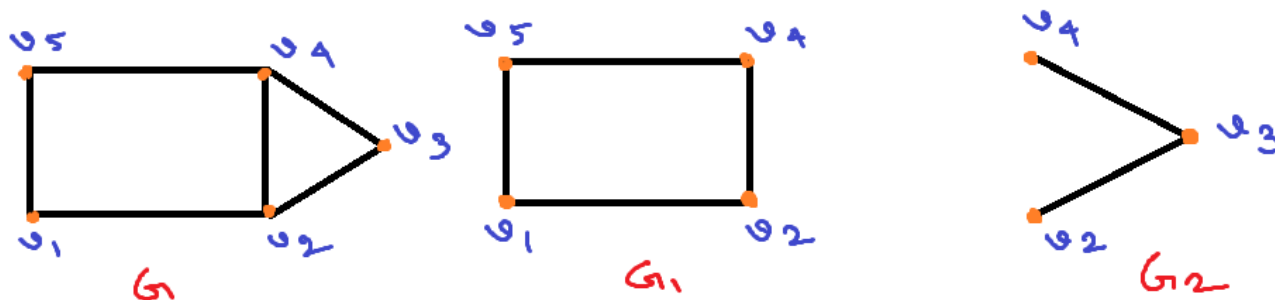


In the figure above  $G_1$  and  $G_2$  are edge-disjoint subgraphs of  $G$

**Note:** Edge disjoint subgraph may have vertices in common but vertex disjoint graph cannot have common edge, so **vertex disjoint subgraph will always be an edge disjoint subgraph**.

If  $G_1, G_2$  be two subgraphs of  $G$ , then  $G$  is said to be **decomposed** into  $\{G_1, G_2\}$

If  $G_1 \cup G_2 = G$  and  $G_1, G_2$  are edge-disjoint.



In the figure above  $G_1$  and  $G_2$  are edge-disjoint subgraphs of  $G$ , and further  $G = G_1 \cup G_2$

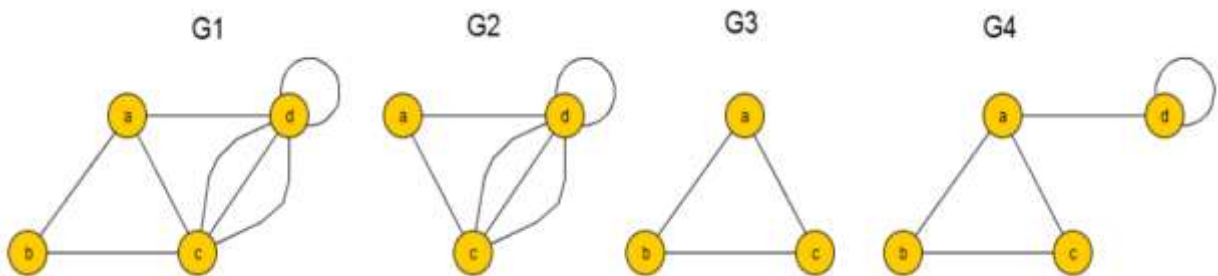
**Problems on Chapter 1:**

1. Does a graph exist with the degree sequence (1,2,2,3,3,4)?
2. Does a graph with 8 edges exist with the degree sequence (1,2,2,3,3,4,5)?
3. Does there exist a simple graph with degree sequence (2,2,3,4,5,6)?
4. Show that there does not exist a regular graph with order 5 and size 9.
5. Find the order of a 3-regular graph with size 12.
6. Show that the maximum number of edges in a simple graph with order  $n$  is  $\frac{1}{2}n(n - 1)$ .
7. Show that there does not exist a simple graph with degree sequence (1,3,3,3).
8. Find the maximum number of vertices in a simple graph with 35 edges and degree of each vertex is at least 3.
9. What is the number of simple graphs possible with 5 vertices?
10. What is the number of simple graph possible with 4 vertices and 3 edges?
11. Show that a complete graph with  $n$  vertices contains  $n(n - 1)/2$  edges.
12. Let  $G$  be a graph of order 20 and size 62. If  $d(v) = 3$  or  $7, \forall v \in V$ , find the number of vertices with degree 3 and degree 7.
13. Given the following graph, draw a ....



- a) A subgraph  $H_1$  with order 7, size 6.
- b) A spanning subgraph  $H_2$  with size 5.
- c) Induced subgraph  $H_3$  with order 4.
- d) Induced subgraph  $H_4$  of size 4.

14. Given the following graph,....

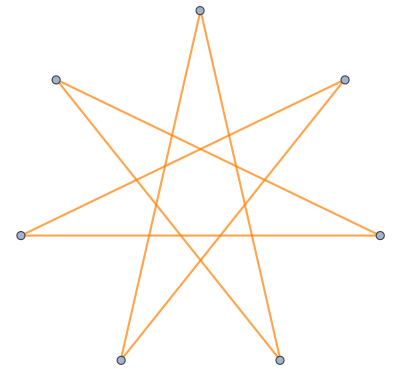
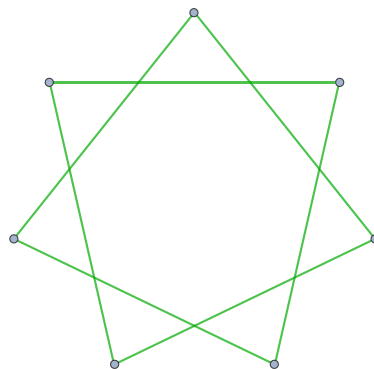
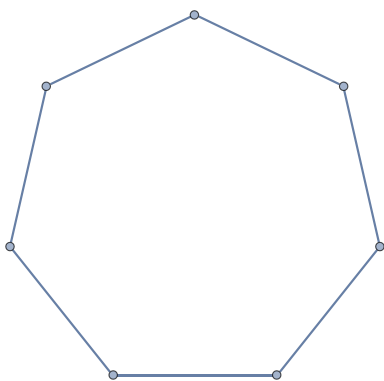


- (a) Find the subgraphs of  $G_1$ ;
- (b) Find the induced subgraphs of  $G_1$ .

15. Decompose the complete graph  $K_7$  into cycles  $C_7$ .

Hints & Solutions:

1. Sum of degrees = 15 (odd) not possible!!!
2. Sum of degrees = 20  $\neq$  2.8 (2 $\times$  no. of edges) not possible!!!
3. For a simple graph  $\max\{d(v)\} = n - 1 = 6 - 1 = 5$ ; 6 is not possible!!!
4. Let it be  $p$ -regular; then  $5p=2.9=18$ , giving  $p = 18/5$  ; not possible!!!
5. Let order be  $n$ ; hence  $3.n=2.12$ , giving  $n=8$ .
6. Since in a simple graph  $\max\{d(v)\} = n - 1 \Rightarrow \sum \max\{d(v)\} = n(n - 1) = 2|E|$ ; hence the result.
7. Let  $d(v_1) = d(v_2) = d(v_3) = 3$  &  $d(v_4) = 1$ ;  
hence it follows that  $v_4$  is adjacent to all the rest; hence  $d(v_4)$  must be 3  $\neq$  1.
8. Since,  $|V|\delta(G) \leq 2|E| \Rightarrow |V| \leq \frac{2|E|}{\delta(G)} = \frac{70}{3} = 23.333..; \Rightarrow |V| = 23$ .
9. Number of edges: Min.= 0 & Max. =  $5(5-1)/2=10$ ; hence edges can range from 0 to 10; Answer is :  
 ${}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + \dots + {}^{10}C_{10} = 2^{10}$
10. Max. edges possible is  $4(4-1)/2=6$ ; Answer is  ${}^6C_3 = 20$
11. Proof by induction: let  $n=1$  then number of edges is  $0=1(1-1)/2$ ;  
let it be true for  $n=k$ ; i.e. if there are  $k$  vertices then edges will be  $k(k-1)/2$ ;  
if we now add a vertex that must connect with all other  $k$  vertices and therefore must add  $k$  edges; so number of edges =  $k(k-1)/2 + k=(k^2+k)/2=(k+1)(k+1-1)/2$ .  
Hence the result follows by induction.
12. Let  $n$  vertices of order 3; so no. of vertices with order 7 is  $20-n$ ;  
So total degree of all vertices= $3n+7(20-n)=140-4n=2|E|=2.62=124$  ; giving  $n=4$ .
13. Do yourself!!
14. (a)  $G_2, G_3$  &  $G_4$  ; (b)  $G_2$  &  $G_3$ , but not  $G_4$  (Why?)
15. The three cycles are as shown below:



[Draw the complete graph  $K_7$  and verify yourself!]

Assignments on Chapter 1:

1. In a group of 9 person, is it possible for each person to shake hands with exactly 5 other persons? Justify your answer.
2. Does there exist a simple graph with degree sequence  $(2,2,2,4,5,5)$ ?
3. Draw a graph with vertices  $v_1, v_2, \dots, v_5$  such that  $d(v_1) = 3, d(v_3) = 2, d(v_2)$  is odd,  $v_4, v_5$  are adjacent.
4. If a graph is regular of odd-degree, show that it has even order.
5. For the graph  $G$  in the following figure, find the induced subgraph  $G[A]$  on the edge set  $A = \{e_1, e_4, e_5, e_6\}$ .

