Bipartite graph (Bigraph)

A graph $G(V,E)$ is called a **Bipartite** if its vertex set $V(G)$ can be partitioned into two non-empty disjoint subsets $V_1(G)$& $V_2(G)$ in such a manner that each edge $e \in E(G)$ has one of its end point in V_1 and the other end point in V_2.

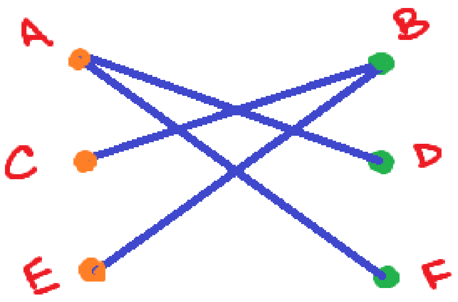


Figure on the left is a bipartite graph, with

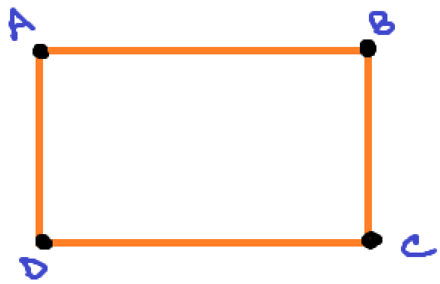
$V_1 = \{A, C, E\}$ and $V_2 = \{B, D, F\}$

Observe that each edge has one end in V_1 and other in V_2

The partition $V = V_1 \cup V_2$ is called a bi-partition of G.

A bipartite graph cannot contain any self-loop.

For a bipartite graph one cannot find an edge whose both ends lie on the same subset of the vertex set, which also means vertices belonging to the same subset cannot be adjacent.



Is the graph on the left bipartite?

Yes!

Here, $V_1 = \{A, C\}$ & $V_2 = \{B, D\}$

Observe that A, C and B, D are not adjacent.

Let $G = (V_1, V_2, E)$ be a bipartite graph

Then, $\sum_{v \in V_1} d(v) = \sum_{v \in V_1} d(v) = |E|$

Result: A bipartite graph does not contain any odd cycle.

Let $G(V_1, V_2, E)$ be a bipartite graph and let $C_k = \{v_1, e_1, v_2, e_2, \dots, v_k, e_k, v_{k+1} = v_1\}$ be a cycle of length k.

Since G is bipartite v_i and v_{i+1} belong to different vertex sets, $\forall i = 1, \dots, k$

Without loss of generality let's assume $v_1 \in V_1$, then $v_i \in V_1$ if i is odd and $v_i \in V_2$ if i is even.

But $v_{k+1} = v_1 \Rightarrow k + 1$ must be odd $\Rightarrow k$ must be even; i.e. C_k must be even.

Note: This result is significant in disproving any graph as bipartite. As any graph that contains an odd cycle is certainly not bipartite.

Result: Maximum number of edges in a Bipartite graph with n- vertices is $\frac{1}{4}n^2$.

Let $G(V_1, V_2)$ with $|V_1| = k,\ |V_2| = n - k$
Since, each vertex in V_1 can connect to a maximum of $n - k$ vertices in V_2; hence
maximum number of edges = $k(n-k)$ which is max. if $k = \frac{n}{2} \Rightarrow max.no.of\ edges = \frac{n}{2}\left(n - \frac{n}{2}\right) = \frac{1}{4}n^2$.

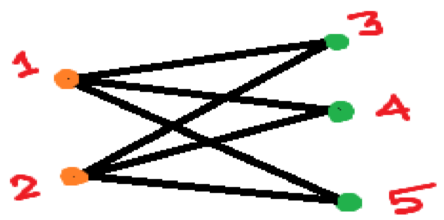
Complete bipartite graph

A bipartite graph $G(V_1, V_2, E)$ is called a **Complete bipartite graph** if every vertex in $V_1(G)$ is joined to every vertex in $V_2(G)$.

If $|V_1| = m$ & $|V_2| = n$, then it is denoted by $K_{m,n}$.

From the definition it is clear that size of $K_{m,n}$ is mn.

Thus $\sum_{v \in V_1} d(v) = \sum_{v \in V_2} d(v) = mn$



In the figure on the left the graph is a Complete Bipartite Graph K_{23}, with $V_1 = \{1,2\}$ and $V_2 = \{3,4,5\}$

All the vertices in V_1 are connected with all in V_2

Size of $K_{23} = 2 \times 3 = 6$

Further examples of complete bipartite graphs:

(a) Complete bipartite graph K_{33} (also called Bi-cubic graph)
(b) Complete bipartite graph K_{43}

Isomorphism of graphs:

Let $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ be two graphs.

An isomorphism is a bijection $f: V_1 \to V_2$ that also preserves adjacency and non-adjacency of vertices and edges between the two graphs.

If v_i, v_j are adjacent in V_1 then $f(v_i), f(v_j)$ are adjacent in V_2. Similarly, if two edges are adjacent in G_1 then they are also adjacent in G_2.

Hence G_1, G_2 are isomorphic, if there exists a one-to-one correspondence $f: V_1 \to V_2$ and a one-to-one correspondence $\phi: E_1 \to E_2$, so that the function maps $v_i, v_j \in V_1$ to the end points of $\phi(e)$ in V_2, $\forall e \in E_1$

i.e. if v_i & v_j are the end points of e in G_1 then $f(v_i)$ & $f(v_j)$ should be the end points of $\phi(e)$ in V_2.

Two graphs G_1 and G_2 are said to be isomorphic if there exists an isomorphism between them, and is denoted as $G_1 \cong G_2$.

Two graphs are isomorphic if they have identical graph-theoretic properties.

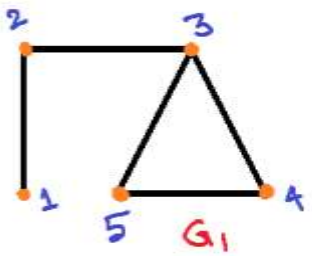
For two isomorphic graphs G_1, G_2, we must have,

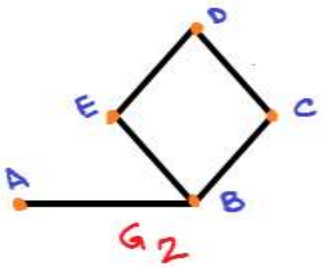
a. $|V_1| = |V_2|$

b. $|E_1| = |E_2|$ and,

c. The degree sequences are the same for both graphs.

Example:





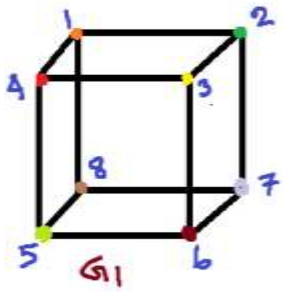
Here, $|V_1| = |V_2| = 5$; $|E_1| = |E_2| = 5$

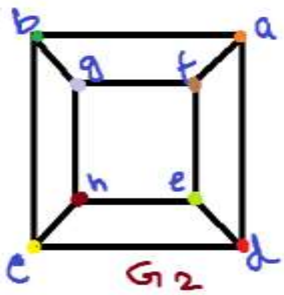
Again, degree sequence of both graphs is given by (1,2,2,2,3)

But still the graphs are not isomorphic!

In G_1 the pendant vertex is adjacent to a vertex of degree 2, whereas in G_2 it is adjacent to the vertex with degree 3 [#1].

Example:





Here, $|V_1| = |V_2| = 8$; $|E_1| = |E_2| = 12$
Also both are 3-regular graphs.

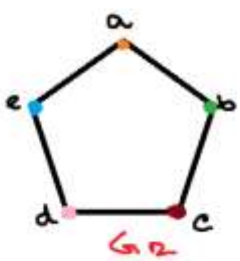
If we define a mapping from G_1 to G_2 as,

$$f(1) = a; f(2) = b; f(3) = c; f(4) = d;$$
$$f(5) = e; f(6) = h; f(7) = g; f(8) = f;$$

Then, it is bijective and it also preserves incidence relation, so that as 1 in G_1 is adjacent to 2,4 and 5, so a in G_2 is adjacent to b, d and e.

Same is true for other vertices as well. Hence in this case $G_1 \cong G_2$.

Example:



Here again, $|V_1| = |V_2| = 5$; $|E_1| = |E_2| = 5$

Both G_1, G_2 are **2-regular** graphs and both are **cycle** graphs

We define a mapping as,

$$f(1) = a; f(2) = d; f(3) = b; f(4) = e; f(5) = c;$$

Then f is bijective and preserves adjacency between vertices [i.e. if two vertices are adjacent in G_1, there f images are adjacent in G_2. Hence $G_1 \cong G_2$.

#1 This example illustrates that if two graphs have the same graph-theoretic properties, that does not necessarily that they are isomorphic! i.e. equality in terms of graph-theoretic properties is a necessary condition but not a sufficient condition for two graphs to be isomorphic.

Walk: An alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it.

Both an edge or a vertex may appear more than once in a walk.

WALK is also known as **CHAIN.**

The ***length of a walk*** is the total number of edges in the walk sequence.

Let $W = \{v_1, e_1, v_2, e_2, \dots, v_n, e_n, v_{n+1}\}$ be a walk.

Then length of $W = n$.

A walk of length 0 is a ***Trivial*** walk, otherwise it is ***Non-trivial*** walk.

When a non-trivial walk begins and ends with the same vertex, it is known as the ***Closed walk***, otherwise it is called an ***Open*** walk.

Trail: A walk with no repeated edges.

A trail of length 0 is a ***Trivial*** trail, otherwise it is ***Non-trivial*** trail.

A trail with different starting and ending vertices is an ***OPEN* trail.**

A trail with same starting and end vertices is a ***CLOSED* trail.**

Path: A trail having no repeated vertices (except possibly the starting and ending vertices).

In a PATH no edge is repeated & no vertex is repeated (except start/end vertex)

A **path** with zero length is ***Trivial***-otherwise ***Non-Trivial***.

Closed path is a path in which **starting & end vertices are same.**

A vertex v in a graph G is ***reachable*** from a vertex u if there exists a ***Path*** from u to v.

Let $G(V, E)$ be a graph of order n and size $n - 1$.

Let $V = \{v_1, \dots, v_n\}$ and $E = \{e_1, \dots, e_{n-1}\}$ s.t. $e_i = \{v_i, v_{i+1}\}, i = 1, .., n - 1$
then G is called a **Path Graph** or **Linear Graph** of order n, denoted by P_n.

There are exactly **Two** pendant vertices in P_n, they are v_1 & v_n, the remaining $n - 2$ vertices all have degree 2.

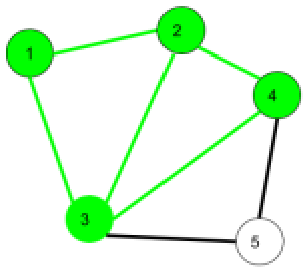
Circuit: A non-trivial closed trail. i.e. a walk with no repeated edges that begins and ends with same vertex.

A non-trivial closed path is a ***cycle***.

A ***cycle graph*** is a single ***cycle***.

A graph containing at least one cycle is a **Cyclic** graph, otherwise it is **Acyclic**.

Example:

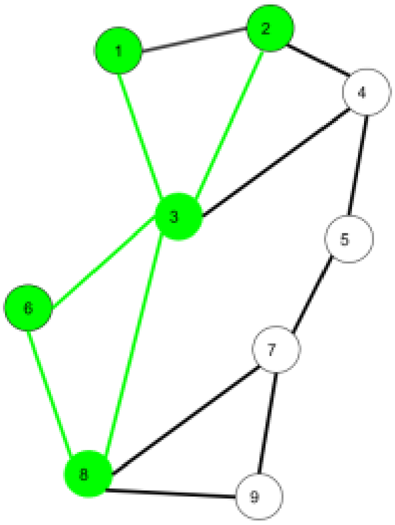


WALK:

A walk is a sequence of vertices and edges of a graph i.e. if we traverse a graph then we get a walk.
Vertex can be repeated
Edges can be repeated

Here 1->2->3->4->2->1->3 is a Closed walk.

Again, 1->2->3->4 is an Open walk.

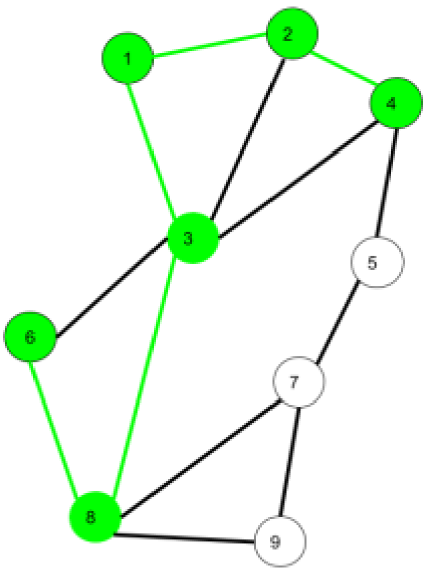


TRAIL:

A Walk in which no edge is repeated then we get a trail
Vertex can be repeated
Edges not repeated

Here 1->3->8->6->3->2 is trail

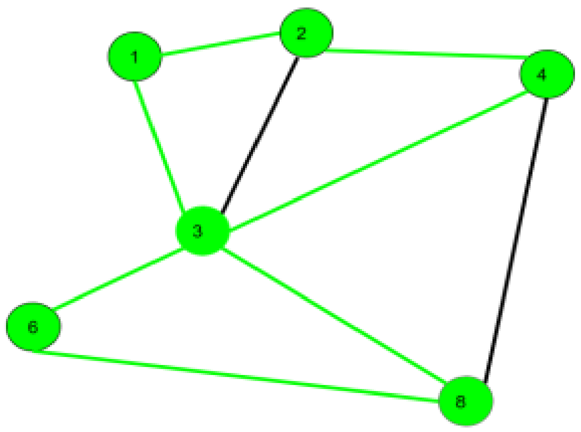
Also 1->3->8->6->3->2->1 will be a closed trail



PATH:

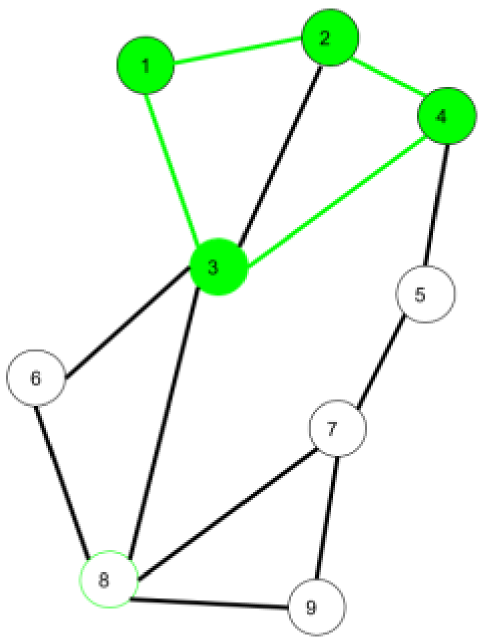
It is a trail in which neither vertices nor edges are repeated i.e. if we traverse a graph such that we do not repeat a vertex and nor we repeat an edge.
Vertex not repeated
Edge not repeated

Here 6->8->3->1->2->4 is a Path



Traversing a graph such that not an edge is repeated but vertex can be repeated and it is closed also i.e. it is a closed trail.
Vertex can be repeated
Edge not repeated

Here 1->2->4->3->6->8->3->1 is a circuit



Traversing a graph such that we do not repeat a vertex nor we repeat a edge but the starting and ending vertex must be same i.e. we can repeat starting and ending vertex only then we get a cycle.
Vertex not repeated
Edge not repeated

Here 1->2->4->3->1 is a cycle.

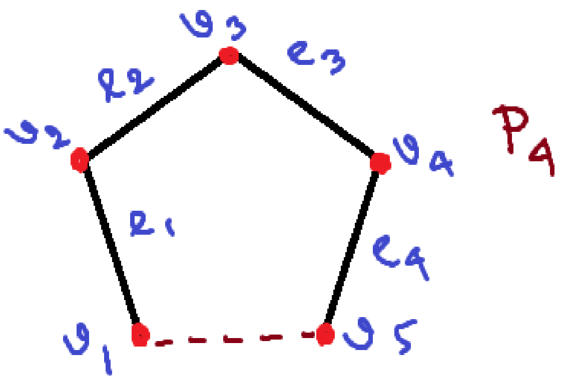
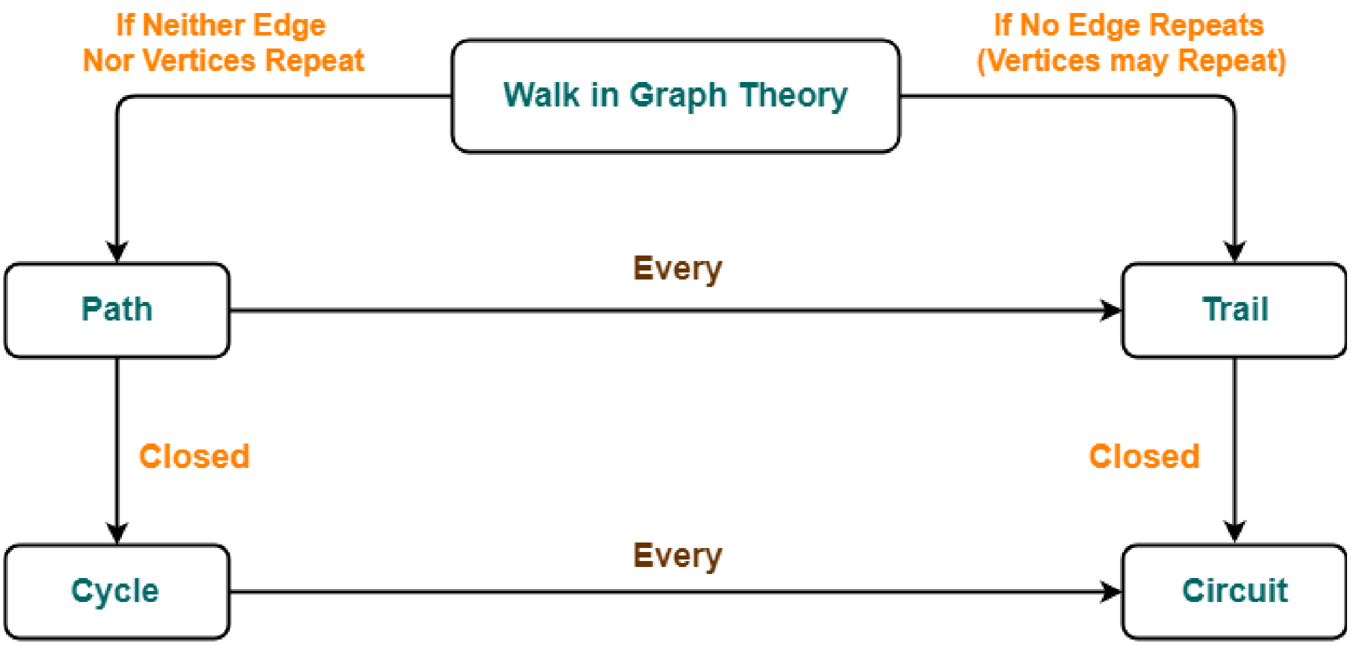


Figure on the left shows the Path Graph P_4

$$\{v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5\}$$

Notice that it can be obtained by edge deletion (dotted line in the graph) from the cycle C_4



Important Chart to Remember

Note 1: Every path is a trail but not conversely, ***AS IN A TRAIL A VERTEX MAY REPEAT***.

Note 2: Every cycle is a circuit but not conversely ***AS IN A CIRCUIT A VERTEX MAY REPEAT***.

Sub-walk: Let $W = (v_1, e_1, v_2, \dots, v_n, e_n, v_{n+1})$ be a walk in a graph. A sub-walk is then a subsequence of W, given by $W_1 = (v_i, e_i, v_{i+1}, \dots, v_j, e_j, v_{j+1})$, where $1 \le i \le j \le n+1$.

Result: Let G be a graph and v, v' be two distinct vertices of G. Then there exists a path between v, v' if there exists a walk from v to v'.

Let, $W = (v = v_1, e_1, v_2, \dots, v_n, e_n, v_{n+1} = v')$ be a walk in G. If W is a path then there is nothing to prove.
If not then $\exists v_i, v_j$ in W such that $v_i = v_j$ [i.e. the vertex v_i is repeated], for $1 < i < j < n$.
Then $\exists$ a closed walk from v_i to v_i itself. Deleting this walk and keeping the vertex intact in W, we get another walk W_1(say) between v, v'. If W_1 is not a path, then we repeat the above process again.
Continuing this, we eventually end up with a path from v to v'.

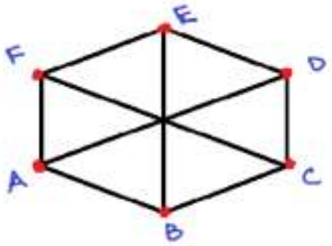
Result: A non-trivial graph G contains a circuit, if each of its vertices has at least degree 2.

Starting with arbitrary $v_1 \in V, d(v_1) \ge 2 \Rightarrow \exists e_1$ that connects v_1 with another vertex v_2; again $d(v_2) \ge 2$; hence by similar argument v_2 is connected with another vertex v_3. This continues until a vertex v_k(say) which was not traversed before. But $d(v_k) \ge 2$; hence $\exists$ at least one edge e_k that connects v_k with a previously traversed node v_i(say).
And thus we have a circuit $(v_i, e_i, v_{i+1}, e_{i+1}, \dots, v_k, e_k, v_i)$.

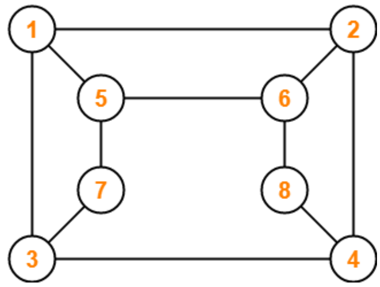
Problems on Chapter 2:

1. Are the following graphs bipartite graphs?

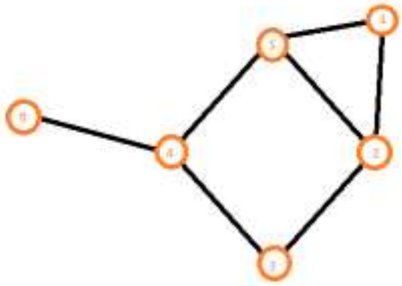
   (a)



   (b)

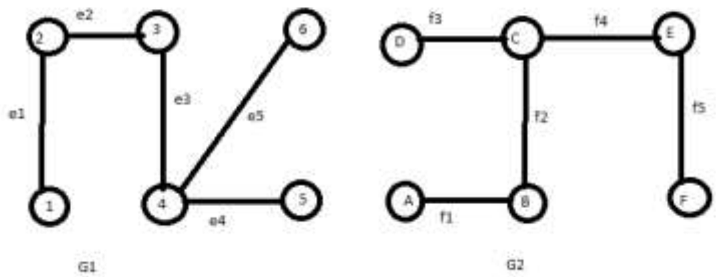


2. Is the given graph bipartite?



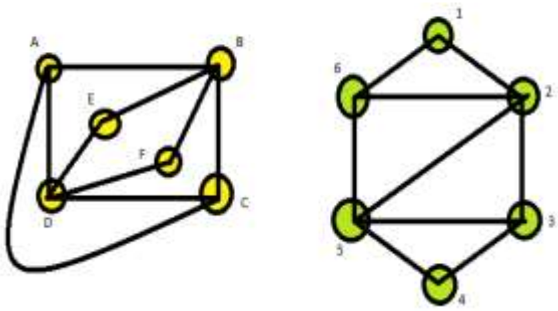
3. Draw a simple graph with 6 vertices, 2 of which have degree 4 and the rest with degree 2.
   (a) Is it bipartite? (b) Is it a complete bipartite graph?

4. Examine if the graphs in the adjoining figure are isomorphic or not.

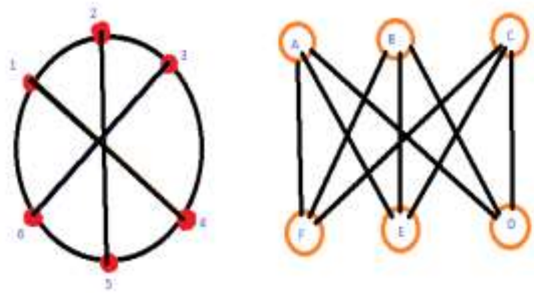


5. Are the following two sets of graphs isomorphic?

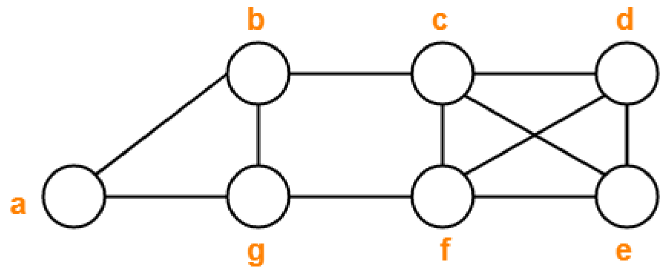
   a.



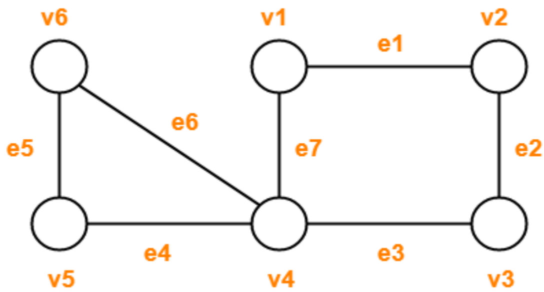
   b.



6. Which of the following sequence of vertices determine walks?
   For those that are walks , decide whether it is a circuit, a cycle, a path or a trail.
   (i) a,b,g,f,c,b ;
   (ii) b,g,f,c,b,g,a ;
   (iii) c,e,f,c ;
   (iv) c,e,f,c,e ;
   (v) a,b,f,a ;
   (vi) f,d,e,c,b ;

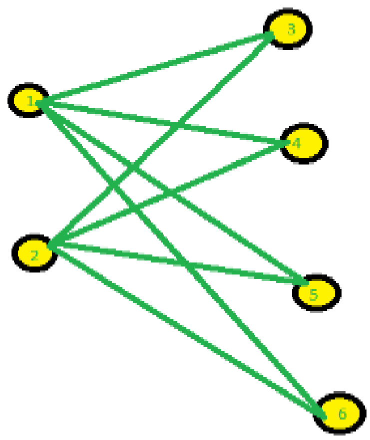


7. Observe the given sequences and predict the nature of walk in each case-
   (i) v1e1v2e2v3e2v2
   (ii) v4e7v1e1v2e2v3e3v4e4v5
   (iii) v1e1v2e2v3e3v4e4v5
   (iv) v1e1v2e2v3e3v4e7v1
   (v) v6e5v5e4v4e3v3e2v2e1v1e7v4e6v6



Hints & Answers:

1. (a) Yes, it is Bi-partite. The two sets are given by $V_1 = \{A, C, E\}$ & $V_2 = \{B, D, F\}$.
   Vertices belonging to the same set are non-adjacent. [**Re-draw the graph with the sets**]
   (b) Yes, it is Bi-partite. The two sets are given by $V_1 = \{1,4,6,7\}$ & $V_2 = \{2,4,5,8\}$.
   Vertices belonging to the same set are non-adjacent. [**Re-draw the graph with the sets**]
2. No, a bi-partite graph cannot contain any odd cycle.
   But in this case the graph contains an odd cycle {1,2,5}.
3.



From the picture it's obvious that it is bi-partite
all the vertices in $V_1 = \{1,2\}$ have degree 4
All the vertices in $V_2 = \{3,4,5,6\}$ have degree 2

Clearly, this is a complete bi-partite graph, as the two vertices on V_1 are connected to all the vertices in V_2.

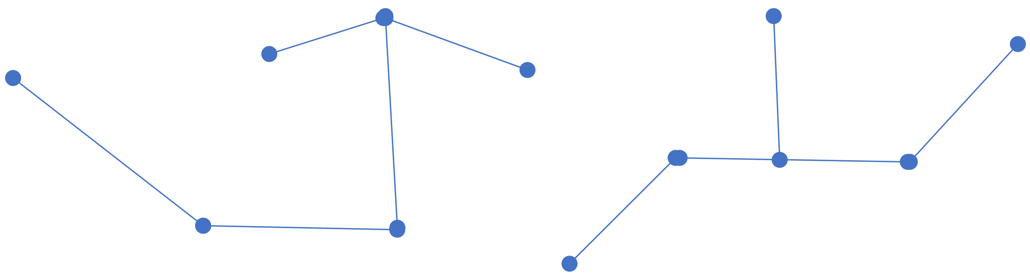
4. **No**. In G_1 the vertex with highest degree is 4 $[d(4) = 3]$; and in G_2 the vertex with highest degree is C $[d(C) = 3]$; hence for isomorphism, 4 must be mapped onto C. But in G_1 vertex 4 is adjacent to two pendant vertices 5 & 6, but in G_2, C is adjacent to one pendant vertex D.

5. (a) **No.** There are two vertices of degree 3 in G_1, A, C and they are adjacent in G_1.
However, in G_2, two vertices of degree 3 are 3 and 6 respectively, but they are non-adjacent.
(b) **Yes.**
Both are bipartite: {1,3,5} & {2,4,6} in G1 and {A,B,C} & {D,E,F} in G2; {1,3,5} can be mapped to {A,B,C} and {2,4,6} can be mapped to {D,E,F} with a mapping of corresponding edges.
[Construct the mapping f]

6. (i) Trail; (ii) Walk; (iii) Cycle; (iv) Walk; (v) Not a walk; (vi) Path;
7. (i) Open walk; (ii) Trail (Not a path because vertex v4 is repeated); (iii) Path; (iv) Cycle; (v) Circuit (Not a cycle because vertex v4 is repeated);

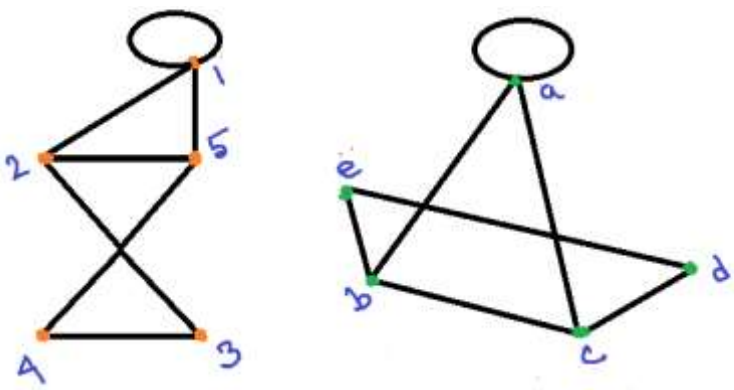
Assignments on Chapter 2:

1.



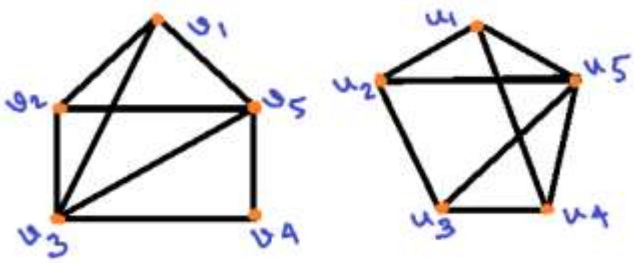
Show that the two graphs above are not isomorphic.

2. Show that a graph with degree sequence (2,2,3,4,4,4,4,5) contains a circuit and a cycle.
3.



Show that the graphs above are isomorphic.

4. Let G be a graph of order $4n$ and size $2m - 1$. Show that G cannot be regular.
5. Find the maximum number of edges in a bipartite graph of order 12.
6.



Examine if the given graphs are isomorphic or not.