Connected Graph:

A graph G is said to be ***connected*** **if there is a path between every pair of vertices** of G.

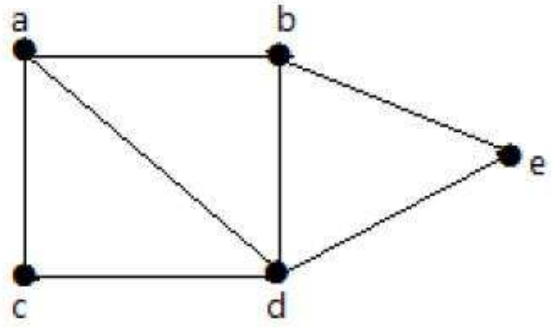
If there is no such path between a pair of vertices, it is called ***Disconnected***.

Let $G = G(V, E)$, and $v_1, v_2 \in V$, then v_2 is said to be ***Reachable*** from v_1 if there exists some path from v_1 to v_2, otherwise it is said to be ***Unreachable***.

Thus, a graph G is connected if there does not exist any unreachable pair of vertices in G.

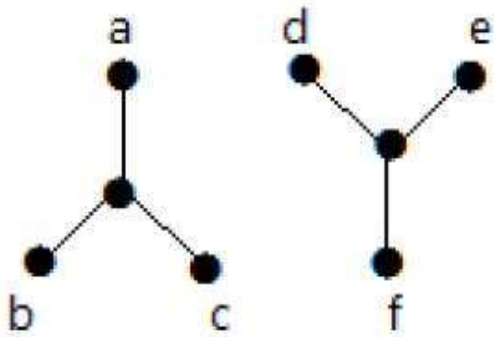
A null graph of order 1 is connected, but, every null graph of order>1 is disconnected.

A connected graph is ***Minimally Connected***, if the removal of any one edge leaves it disconnected.



Graph G_1

Connected Graph: all the vertices are reachable from all other vertices



Graph G_2

Disconnected Graph: vertex d is not reachable from vertex $a, b\ or\ c$

Is the graph G_1 minimally connected?
No, as there does not exist any edge, deletion of which will make it disconnected.

The subgraph G1 (not a null graph) of a disconnected graph G is a ***component*** of G if
1. G_1 is connected and 2. G_1 has no connected supergraph.

Every disconnected graph contains two or more components. In the disconnected graph G_2 above there are two components with vertex sets $V_1 = \{a, b, c\}$ and $V_2 = \{d, e, f\}$.

Consider any vertex v_i in a disconnected graph G. By definition, not all other vertices are connected to v_i by paths. The vertex v_i and all other vertices in G that are connected to it, along with all the edges incident on them, together form a component of G.

Number of components of a graph G is denoted by $\omega(G)$.

For a connected graph clearly $\omega(G) = 1$.

For a null graph of order $n(> 1), \omega(G) = n$

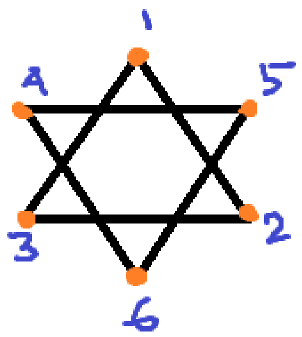
In the graphs G_1 & G_2 above, $\omega(G_1) = 1$ & $\omega(G_2) = 2$

Result: A graph $G(V, E)$ is disconnected iff its vertex set V can be partitioned into two non-empty disjoint subsets V_1 & V_2 such that there exists no edge in E with one end vertex in V_1 and other in V_2.

First suppose that such a partition exists. Consider two arbitrary vertices $a, b \in G$, such that $a \in V_1$ & $b \in V_2$. No path exists between a, b otherwise there would be at least one edge with one end in V_1 and other in V_2. Hence G is disconnected.

Conversely, let G be a disconnected graph. Let V_1 be the set of all vertices in G that are connected to a (i.e. there exists some path between a vertex in V_1 and a). Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form a non-empty set V_2. No vertex in V_1 is joined to any in V_2 by an edge, and also $V_1 \cup V_2 = V$; hence the partition is defined.

Example:



In the graph on the left, let $V_1 = \{1,2,3\}$& $V_2 = \{4,5,6\}$

Then $V_1 \cup V_2 = V$ & $V_1 \cap V_2 = \phi$

Also there does not exist any edge with one end in V_1 and other in V_2.

Hence the given graph is disconnected

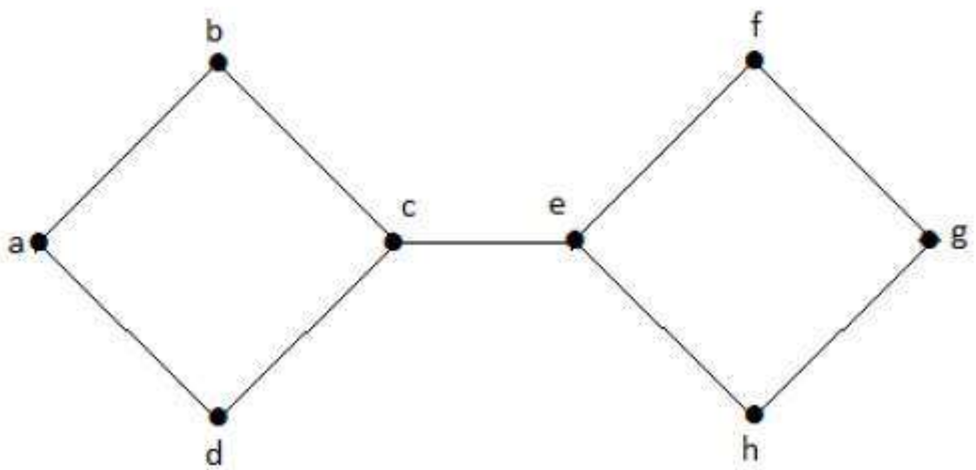
Here, $\omega(G) = 2$

Cut Vertex

Let G be a connected graph. A vertex $v \in$ G is called a cut vertex of G, if G-v (Delete v from G) results in a disconnected graph. Removing a cut vertex from a graph breaks it in to two or more components.

In the following graph, vertices e and c are the cut vertices.

Example:



By removing e or c, the graph will become a disconnected graph.

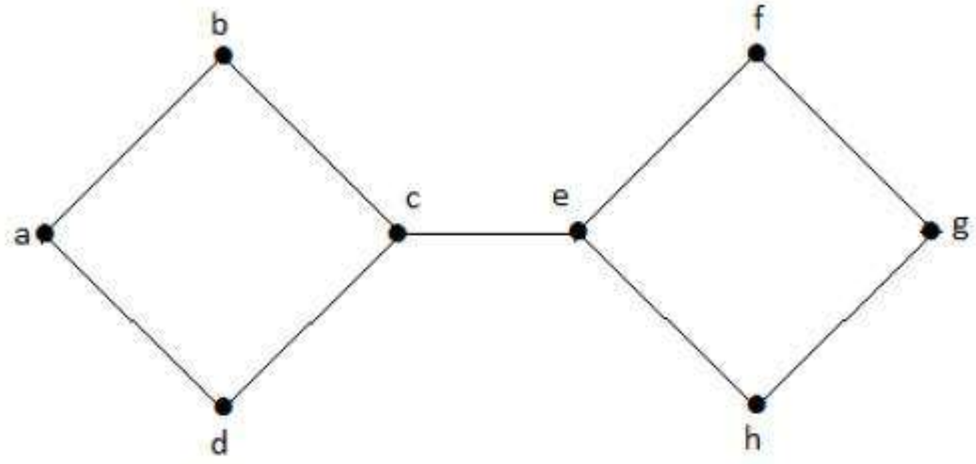
Cut Edge (Bridge)

Let G be a connected graph. An edge e ∈ G is called a cut edge if G-e results in a disconnected graph.

If removing an edge in a graph results in to two or more graphs, then that edge is called a Cut Edge.

Example:

In the following graph, the cut edge is [(c, e)]



By removing the edge (c, e) from the graph, it becomes a disconnected graph.

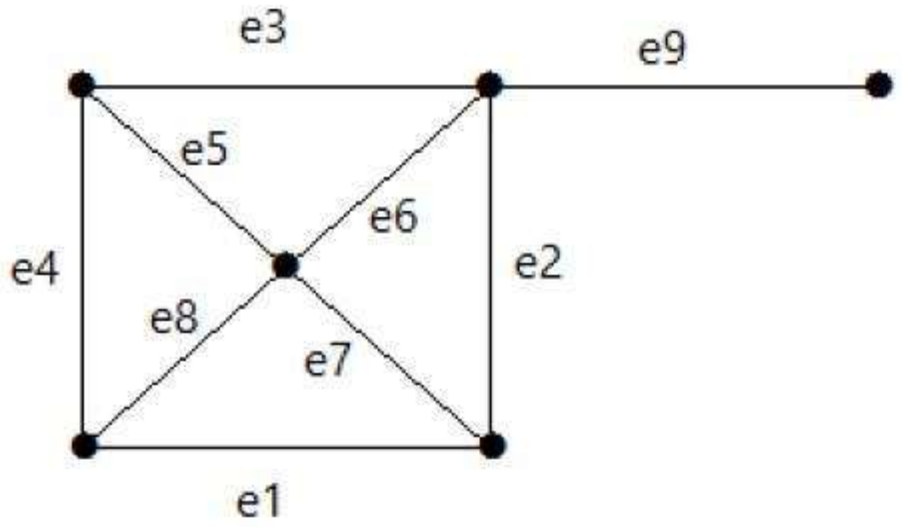
Cut Set of a Graph

Let G=G (V, E) be a connected graph. A subset E' of E is called a cut set of G, if deletion of all the edges of E' from G makes G a disconnected graph.

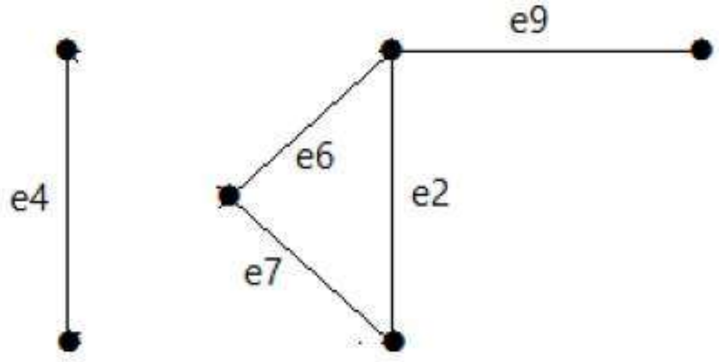
In other words, if deleting a certain number of edges from a graph makes it disconnected, then those deleted edges are called the cut set of the graph.

Example:

Look at the following graph. One of its cut set is E1 = {e1, e3, e5, e8}.



After removing the cut set E1 from the graph, it would appear as follows:



Similarly, there are other cut sets that can disconnect the graph:

E2 = {e9}, which is the **smallest cut set** of the graph.

E3 = {e3, e4, e5}

Edge Connectivity

Let G be a connected graph. The minimum number of edges whose removal makes G disconnected is called edge connectivity of G.

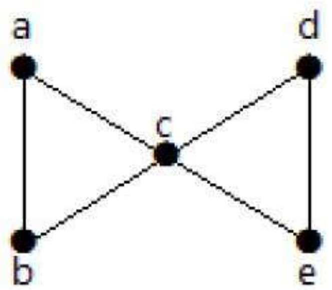
Edge connectivity is denoted as λ(G).

In other words, the **number of edges in a smallest cut set of G** is called the edge connectivity of G.

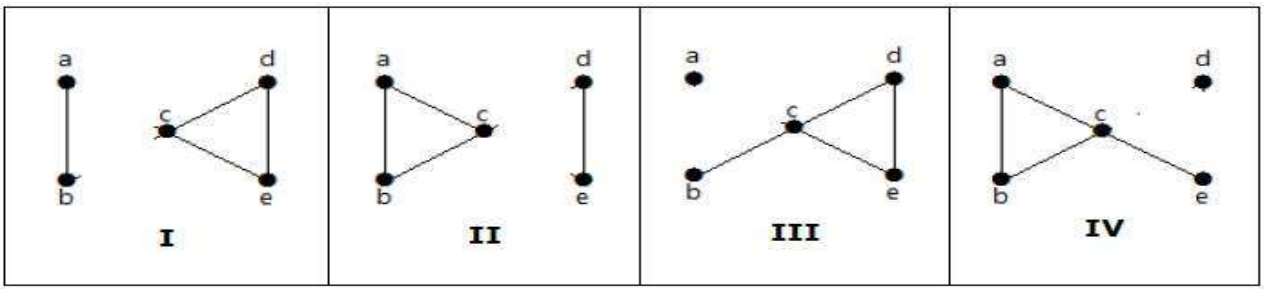
If G has a cut edge, then obviously *λ(G)* = 1 (i.e. edge connectivity of G is 1).

Example:

In the following graph, by removing two minimum edges, the connected graph becomes disconnected. Hence, its edge connectivity (λ(G)) is 2.



Here are the four ways to disconnect the graph by removing two edges:



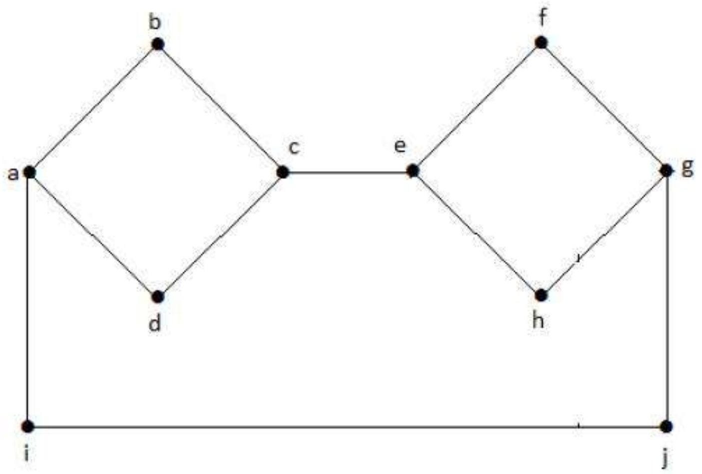
Vertex Connectivity:

Let G be a connected graph. The minimum number of vertices whose removal makes G either disconnected or reduces G in to a trivial graph is called its vertex connectivity.

Edge connectivity is denoted as $\kappa(G)$.

Example:

In the above graph, removing the vertices e and i makes the graph disconnected. Hence, it's vertex connectivity $(\kappa(G))$ is 2.

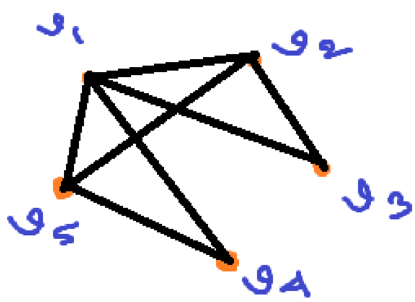


If G has a cut vertex, then obviously, $\kappa(G)$ = 1.

Result: If a graph (connected or disconnected) has exactly two vertices of odd degree, then there must be a path joining these two vertices.

Assume G is a finite graph with exactly 2 vertices of odd degree, u and v (say).
If G is connected, then there is nothing to prove.
[∵ in a connected graph there is a path between any two vertices]
Thus we can assume that G is disconnected. Let J be the connected component of G such that u∈J.
Since J is a graph, it must contain an even number of vertices of odd degree, since, the sum of degrees of all vertices in J must be even (equal to twice the number of edges of J).
Thus, there is at least one more vertex of odd degree in J.
Since G contains exactly two vertices of odd degree, we must have v∈J.
This implies that there is a path from u to v.

Example: Let us construct a graph with degree sequence (4,3,3,2,2):



In the graph

$d(v_1) = 4; d(v_2) = d(v_5) = 3; d(v_3) = d(v_4) = 2$

Therefore, the odd degree vertices are v_2 & v_5

Clearly there is a path between them.

Note: As the graph is connected, there is a path between any two vertices.

Result: A simple graph with n vertices and k components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges.

The maximum number of edges is clearly achieved when all the components are complete.

Moreover, the maximum number of edges is achieved when all of the components except one have one vertex each.
The proof is by contradiction. Suppose the maximum is achieved in another case in which there exist two components with more than one vertex; say, the number of vertices for the two components be p and $q(p \geq q)$.
Let us pick the component with q vertices.
Take one of its vertices and delete it, thereby removing $q-1$ edges.
Now add a new vertex to the component with p vertices and join it to all other vertices, adding p edges.
Clearly then this graph has more edges (as $p \geq q$), contradicting the maximality of the graph.
Hence the maximum is achieved when only one of the components has more than one vertex.
Thus, we have $k-1$ components each with one vertex and zero edges,
and 1 component with $n-(k-1)=n-k+1$ vertices.
The number of edges of this component will be maximum when it is complete; i.e. each vertex is adjacent to all other vertices of this component, and for a complete graph with $n-k+1$ vertices, the number of edges will be $= \frac{(n-k+1)(n-k+1-1)}{2} = \frac{(n-k)(n-k+1)}{2}$ $\left[\because no. of\ edges\ in\ K_n\ is \frac{n(n-1)}{2}\right]$.
Since all other components have zero edges, hence the total maximum number of edges for the entire graph will be $= \frac{(n-k)(n-k+1)}{2}$.

Example: Let G be a simple graph with order 9 and 6 components. Find the maximum size of G.

Here, $n=9\,; k=6$

The maximum size of $G = \frac{(n-k)(n-k+1)}{2} = \frac{(9-6)(9-6+1)}{2} = \frac{3.4}{2} = 6$.

Result: The size of every connected graph of order n is at least $n-1$.

Assume that there exists a complete graph G of n vertices. Hence the no of edges $= \frac{n(n-1)}{2}$

Now any complete graph of n vertices is an (n-1) regular graph (Meaning that the degree of each vertex is n-1. i.e., there are n-1 edges incident to each vertex of a complete graph).

Now we need the minimum no of edges but we cannot remove all edges as the graph needs to be connected. Hence the best possible way is to remove n-2 edges from the first vertex. Now the degree of all vertices that the removed edges are incident on is reduced by one except for the one vertex which is incident on by the edge we did not remove from the first vertex.

Now we can remove n-3 edges from the next vertex while not disconnecting it and n-4 from the next and so on.

Hence the total no of edges that needs to be removed from the graph in order to get the minimum no of edges while not disconnecting the graph $= (n-2)+(n-3)+\cdots+3+2+1 = \frac{(n-2)(n-1)}{2}$

The remaining graph is connected with minimum no of edges. Hence the no of edges remaining $\frac{n(n-1)}{2} - \frac{(n-2)(n-1)}{2} = n-1$

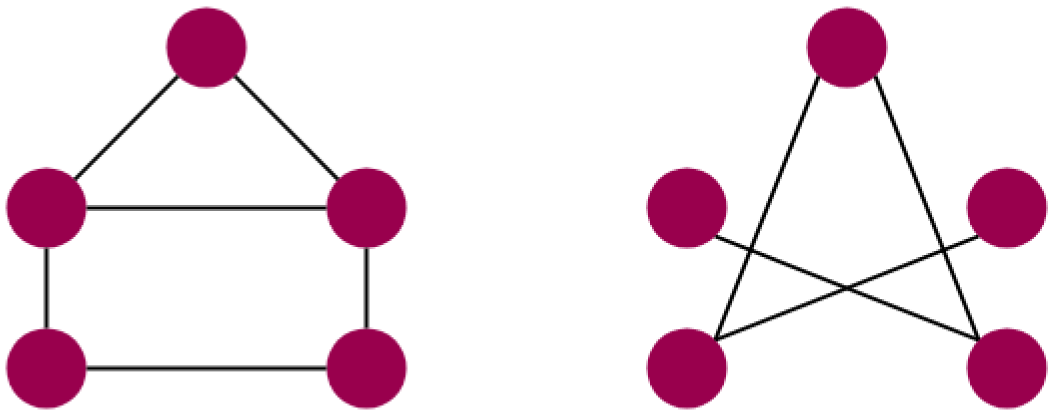
Complement Graphs

If a graph G has n-vertices then the complement $\bar{G}$ is the complete graph K_n with all of the edges in G deleted from it. It is also known as the inverse of the graph G.

We can see this very clearly in the following example showing both the graph G and its complement:

G $\qquad$ $\bar{G}$ (Complement of G)

Further example:



Graph G $\qquad$ Complement Graph $\bar{G}$

The following relationship exists between a graph G and its complement graph $\bar{G}$:

(1) Number of vertices in G = Number of vertices in $\bar{G}$

In the first example above, $V(G) = V(\bar{G}) = 4$; in the second $V(G) = V(\bar{G}) = 5$

(2) The sum of total number of edges in G and G' is equal to the total number of edges in a complete graph with the same vertices.

In the example 1 above $E(G) + E(\bar{G}) = 3 + 3 = 6 = E(K_4)\left[no. of\ edges\ in\ K_4 = \frac{4(4-1)}{2} = 6\right]$

In the example 2 above $E(G) + E(\bar{G}) = 6 + 4 = 10 = E(K_5)\left[no. of\ edges\ in\ K_5 = \frac{5(5-1)}{2} = 10\right]$

In the example 1 above we can see that G is disconnected but the complement $\bar{G}$ is connected.

Result: Any simple graph G and its complement cannot both be disconnected.

Suppose G is disconnected, say $G_i, i = 1, .., m$ be the connected components of G.

Let us take any two vertices $v_1, v_2 \in \bar{G}$,

Case 1: if v_1, v_2 are in the same connected component of G (say, G_i), then in the graph $\bar{G}$, we will get an edge from v_1 to some vertex of a different path component (say, G_j) and from there we will get an edge to v_2 and hence they are connected in $\bar{G}$.

Case 2: If two vertices are in different path components of G, then they are clearly connected in $\bar{G}$.

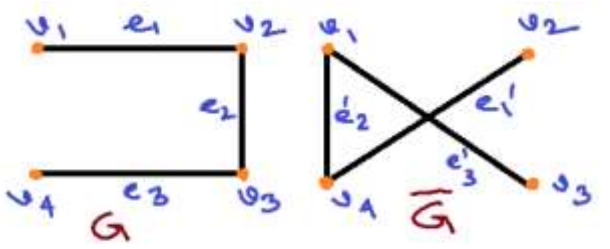
Thus, in any case the two vertices are connected in $\bar{G}$ and hence the result follows.

Self-complementary graph

A simple graph G is self-complimentary if it is isomorphic to its own complement.

i.e. $G \cong \bar{G}$

Example:



In the figure on the left,

$f(v_1) = v_2\,; f(v_2) = v_4; f(v_3) = v_1\ \&\ f(v_4) = v_3$

Clearly there is a bijection that also preserves the adjacency of vertices

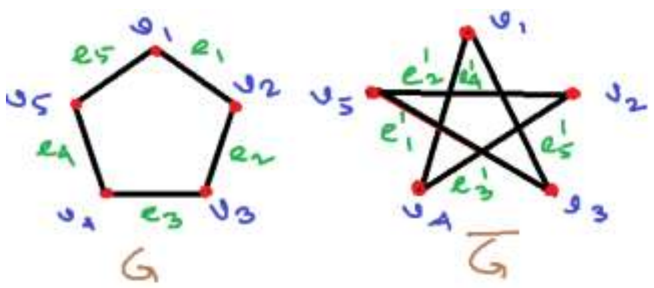
Hence, $G \cong \bar{G}$

Result: Any self-complementary graph has to have 4k or 4k+1 vertex, for some k∈N.

If G is self-complementary, then its edges contain half of all of the possible edges (and the complement has the others).

Thus, if G has n vertices, its number of edges is $\frac{n(n-1)}{4}$ $\left[\because\ order\ of\ K_n = \frac{n(n-1)}{2}\right]$

Thus, n or n−1 must be a multiple of 4. That is, n=4k or n=4k+1, for some $k \in \mathbb{N}$.



Here, let us consider

$f(v_1) = v_3; f(v_2) = v_5; f(v_3) = v_2;$
$f(v_4) = v_4\ \&\ f(v_5) = v_1$

Then clearly it is a bijection and also the adjacency of vertices is preserved.

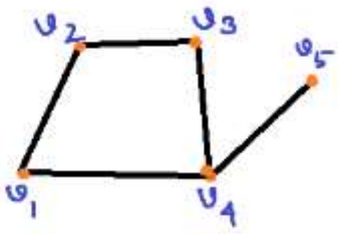
Hence, $G \cong \bar{G}$

In both the above examples one can see the number of vertices is 4 (in 1st Example) or 5 (in 2nd Example) i.e. either it is $4k$ or $4k$+1, $k \in \mathbb{N}$

Problems on Chapter 3:

1. Let G be a connected graph of order $n \geq 2$ and size $m < n$. Prove that G has at least one pendant vertex.

2.



Examine if the graph on the left is self-complimentary or not.

3. Show that the total number of edges incident on any two vertices in the complete graph K_n is $2n - 3$.

4. If in a disconnected graph G a vertex v is connected to a vertex v' belonging to a component G', then prove that v is also a vertex of G'.

5. For a connected graph G show that, $\kappa(G) \leq \lambda(G) \leq \delta(G)$, where $\kappa(G)$ is the vertex connectivity, $\lambda(G)$ is the edge connectivity and $\delta(G)$ is the minimum degree of G.

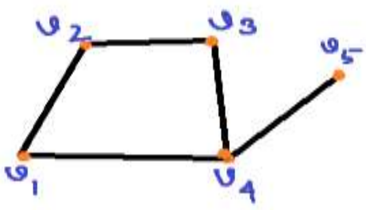
6. A simple graph G has 30 edges and its complement graph G' has 36 edges. Find number of vertices in G.

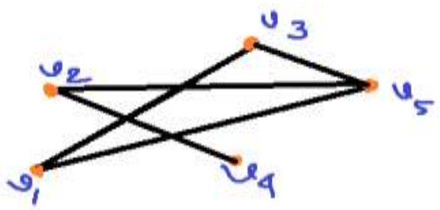
7. Let G be a disconnected graph of order n and size m. If G has exactly k components, show that $m \geq n - k$.

8. Show that if every component of a graph is bipartite, then the graph itself is bipartite.

Hints & Answers:

1. Let ∃ no pendant vertex. Then $d(v) \geq 2, \forall v \in V$; hence $\sum_{v \in V} d(v) \geq 2n$ $[\because total\ vertices = n]$
   Again, $\sum_{v \in V} d(v) = 2|E| \Rightarrow |E| \geq n$; but $|E| = m < n$; Contradiction!!
   Hence at least one vertex must be a pendant vertex (degree=1).
2.





In order to form an isomorphism, we must have, $f(v_5) = v_4; f(v_4) = v_5$

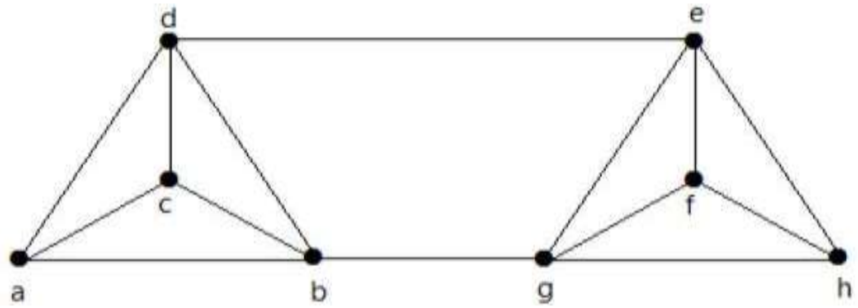
But, in G, the vertex with highest degree i.e. v_4 is adjacent to the pendant vertex v_5, whereas in $\bar{G}$, the vertex with highest degree i.e. v_5 is not adjacent to the pendant vertex v_4.
Hence it is not self-complimentary.

3. Let u, v be any two vertices of K_n. Then $d(u) = n-1; d(v) = n-1$
   Hence total degrees of both $u, v = 2(n-1) = 2n-2$
   But in K_n, u, v are connected by an edge, which is counted twice in the above calculation.
   Hence the actual count would be $2n-2-1 = 2n-3$.
4. If g is connected to v' then there is a path from v to v'.
   Let the path be $v = v_0, e_0, v_1, e_1, \dots, e_{k-1}, v_k = v'$
   Now, $v_k = v'$ is adjacent to v_{k-1} by the side e_{k-1} and hence must belong to the same component G', again v_{k-1} is adjacent to v_{k-2} by the side e_{k-2} and hence must belong to G'.
   Proceeding in this manner we can see that $v = v_0$ must belong to G'.
5. Consider a graph G, and say its edge connectivity is p. Then, there is a cut set E_c with p edges.
   Let's assume that the vertex connectivity is greater than p.
   Arbitrarily choose one end point of each edge in E_c. Call this new set of endpoints V_{E_c}. There's one for each edge, so $|V_{E_c}| = p$.
   Since deleting a vertex means deleting every edge that touches it, so deleting every vertex in V_{E_c} must result in deleting every edge in E_c and thereby making the graph disconnected.
   But here we only deleted p different vertices! Contradicting the fact that vertex connectivity$> p$.
   Therefore, the vertex connectivity can never be greater than the edge connectivity.
   Similarly for the second part, if we remove number of edges equal to the minimum degree of the graph, then the vertex with the minimum degree gets isolated thereby making the graph disconnected and hence edge connectivity cannot exceed the minimum degree of a graph.

6. We know that, $E(G) + E(\bar{G}) = E(K_n) = 30 + 36 = 66$
   Now, a complete graph K_n must have $\frac{n(n-1)}{2}$ edges;
   therefore, $\frac{n(n-1)}{2} = 66 \Rightarrow n^2 - n - 132 = 0 \Rightarrow (n-12)(n+11) = 0 \Rightarrow n = 12$ $[\because n \in \mathbb{N}]$
7. Let the k components be $G_1, G_2, \dots, G_k$ with number of vertices $n_i, i = 1, \dots, k$ and number of edges $m_i, i = 1, \dots, k$
   Since each G_i is connected, it must have a minimum of $n_i - 1$ edges [**See Results**].
   Hence, $m_i \geq n_i - 1, i = 1, \dots, k$
   Then, $\sum_{i=1}^{k} m_i \geq \sum_{i=1}^{k} (n_i - 1) \Rightarrow m \geq n - k$ $[\because \sum m_i = m \,\&\, \sum n_i = n]$
8. Let G has components $G_i, i = 1, \dots, k$. Since G_i is bipartite its vertex set can be separated into sets $A_i \,\&\, B_i, i = 1, \dots, k$. Let $A = \cup_i A_i$ & $B = \cup_i B_i$; then $V(G) = A \cup B \,\&\, A \cap B = \phi$; Hence the result.

Assignments on Chapter 3:

1. A self-complementary has size 150. Find its order.
2. Draw a connected graph of order 5 and size 7, whose complement is disconnected.
3. Calculate $\lambda(G)$ and $\kappa(G)$ for the following graph:



And hence verify the inequality, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

4. Let G be a disconnected graph of order 21 and G_1, G_2 be two components. If order of G_1 is 13, find the minimum possible size of G.
5. If G be a graph with three components, find the maximum possible size of G.
6. Prove that a simple graph with n vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.
7. Draw a simple graph G with degree sequence (3,3,3,3,4). Examine whether G is connected or not. Is $\bar{G}$ connected?
8. Draw a connected graph of order 5 such that the graph remains connected even after the removal of any two of its vertices.