

**Euler Trail:** A trail in a connected multigraph  $G$  is an Euler trail, if it includes all the edges of  $G$ .

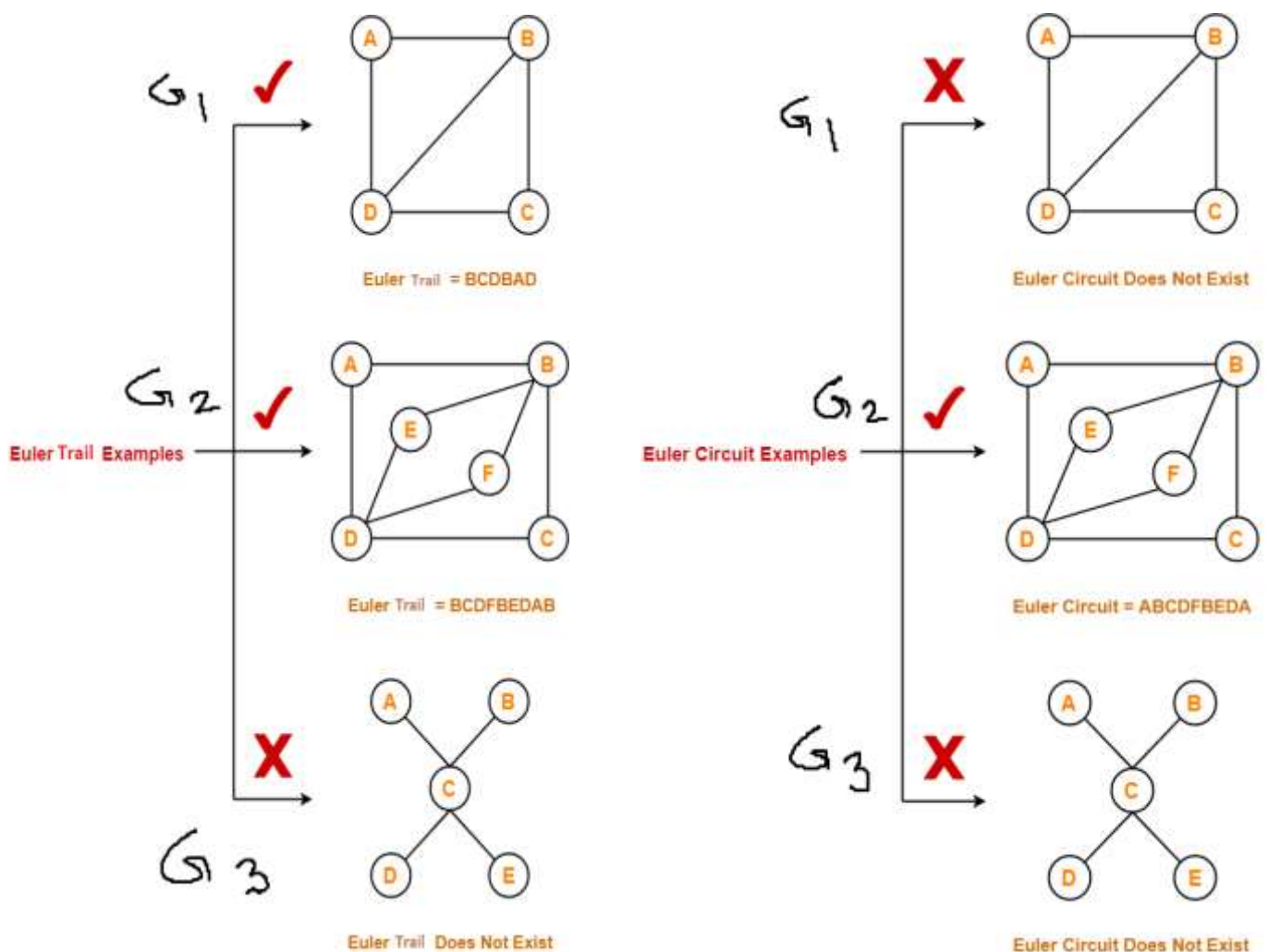
**Semi-Eulerian Graph:** A connected multigraph  $G$  is called Semi-Eulerian (or, *Traversable*) if it contains an Euler trail.

**Euler Circuit:** If an Euler trail in a graph  $G$  is closed, it is an Euler circuit (also known as *Euler Tour / Euler Cycle*).

Thus, a circuit in  $G$  is said to be an Euler circuit, if it includes each edge of  $G$  exactly once.

**Eulerian Graph:** A connected multigraph  $G$  containing an Euler circuit is called an Eulerian graph / *Euler graph*.

**Examples:**



In the examples above:

$G_1$  is a Semi-Eulerian graph as it contains an Euler Path, but does not contain any Euler circuit.

$G_2$  is an Euler graph as also a Semi-Eulerian graph, as it contains both an Euler trail and circuit.

$G_3$  is neither a Semi-Eulerian graph, nor an Eulerian graph.

**Note 1:** From the examples above and from definition, it is clear that every Euler graph is also Semi-Eulerian but not conversely.

**Note 2:** An Eulerian graph or a Semi-Eulerian graph is known as an Edge-traceable graph because it can be drawn with a pencil, without any repetition of edges and without lifting the pencil from the paper.

**Result:** An undirected multi graph  $G$  has an Eulerian circuit (i.e.  $G$  is Eulerian) if and only if it is connected and all its vertices are of even degree.

Let,  $G = (V, E)$  be an Eulerian graph.

**Claim: The degree of each vertex is even.**

As  $G$  is an Eulerian graph, it contains an Eulerian circuit, say  $C$ , which in particular is a closed trail.

Let this walk start and end at the vertex  $u \in V$ .

Since,  $C$  traverses each edge exactly once, each visit of  $C$  to an intermediate vertex  $v \in C$  contributes two (first to reach  $v$  and then go out from  $v$ ) to the degree of  $v$ . So,  $d$  is even for every such vertex.

Again, each intermediate visit to  $u$  contributes two to  $d(u)$  and also the initial and final edges of  $C$  contribute one each to  $d(u)$ . Hence,  $d(u)$  is also even.

Conversely, suppose  $G$  is a connected graph and degree of each vertex of  $G$  be even.

**Claim:  $G$  has an Eulerian circuit.**

We prove the result by induction on  $m$ , the number of edges of  $G$ .

Note that the result is clearly true if  $m = 3, 4$ .

So, let the result be true for any graph that has fewer than  $m$  edges and let  $G = (V, E)$  be a graph with  $m$  edges.

As  $d(v) \geq 2, \forall v \in V$ ,  $G$  contains a cycle. Let  $C$  be a closed trail in  $G$  of maximum length.

If  $C$  contains all the edges of  $G$ , we are done.

If not, consider the graph  $G - E(C)$ , which is a non-trivial even degree graph.

Let  $C_1$  be one of the non-trivial components of  $G - E(C)$ .

Then,  $C_1$  is an even connected graph having less number of edges than  $G$ .

Thus, by induction hypothesis,  $C_1$  is Eulerian with, say  $C_1'$ , as its Eulerian circuit.

Moreover,  $C_1$  has a vertex  $v$  in common with  $C$  [ $C_1$  is obtained by removing only edges from  $G$  and no vertices have been removed].

Hence, we have obtained a circuit  $C_1 \cup C$  which starts and ends at the vertex  $v$ . Also, the length of  $C_1 \cup C$  is strictly larger than that of  $C$ . This contradicts our assumption that  $C$  was a closed trail of maximum length. Hence,  $C$  actually contains all edges of  $G$  and therefore  $G$  is indeed Eulerian.

**Corollary:** The complete graph  $K_n$  is Eulerian if and only if  $n$  is odd.

In a complete graph  $K_n$ ,  $d(v) = n - 1, \forall v$ . The graph will be Eulerian iff  $d(v) = \text{even}, \forall v$  hence,  $n - 1$  must be even  $\Rightarrow n$  must be odd.

**Corollary:** All cycle graphs are Eulerian.

In a cycle graph  $d(v) = 2, \forall v$ , which is even and hence the result.

**Corollary:** The complete bipartite graph  $K_{m,n}$  is Eulerian if and only if both  $m, n$  are even.

In this case  $d(v) = m$  or  $n$ , hence if both  $m, n$  are even, the result follows.

**Result:** A connected multi-graph  $G$  is semi-Eulerian if and only if there are either zero or two vertices of odd degree.

Let  $G$  be a non-directed multi graph. If  $G$  has an Eulerian trail then clearly  $G$  is connected. By definition, the end vertices of this trail, say  $u_0$  and  $u_k$ , have odd degrees and the rest of the vertices have even degrees<sup>#1</sup>. If  $u_0 = u_k$ , then  $G$  has no vertex of odd degree, else,  $u_0, u_k$  are the only vertices of odd degree.

Conversely, let  $G$  be a connected graph having either zero or two vertices of odd degree. If  $G$  has no vertex of odd degree, then by the previous result we find that  $G$  is Eulerian and thereby also Semi-Eulerian.

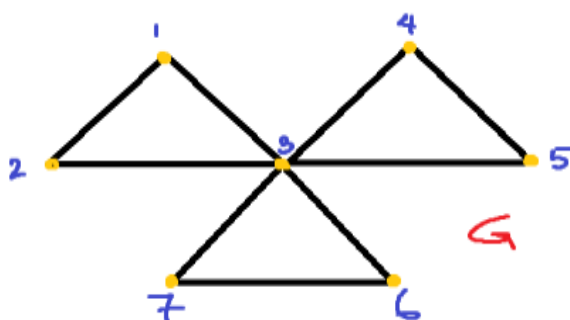
If  $G$  has exactly two vertices of odd degree then consider the new graph  $G' = G + e$ , which is obtained from  $G$  by joining the two odd vertices by an edge  $e$ . Then all the vertices of  $G'$  are even and hence by the previous result will contain an Eulerian circuit. From this circuit if we deduct the edge  $e$ , we are left with an Eulerian trail and hence  $G$  is Semi-Eulerian.

**Result:** A connected graph  $G$  is Eulerian if and only if its edge set can be decomposed into cycles.

Let  $G = (V, E)$  be a connected graph and suppose that  $G$  can be decomposed into cycles. If  $k$  of these cycles are incident at a particular vertex, say  $v$ , then  $d(v) = 2k$ . Therefore, the degree of every vertex of  $G$  is even and hence it is Eulerian.

Conversely, let  $G$  be Eulerian. To show that  $G$  can be decomposed into cycles. To prove this, we use induction on the number of edges of  $G$ . So, the induction hypothesis is that all Eulerian graphs with edges  $< |E(G)|$  can be decomposed into disjoint cycles. Now, since  $d(v) \geq 2, \forall v \in V(G)$ ,  $G$  has a cycle, say  $C$ . Then  $E(G) - E(C)$  is possibly a disconnected graph, each of whose components  $C_1, C_2, \dots, C_k$  is an even degree graph and hence Eulerian. By the induction hypothesis, each  $C_i$  is a disjoint union of cycles. These together with  $C$  provides a partition of  $E(G)$  into cycles.

**Example:**



In the picture, the graph  $G$  can be decomposed into three cycles  $(1,2,3)$ ,  $(3,4,5)$  and  $(3,6,7)$

Also, it can be easily seen that  $G$  is Euler. [degree of each vertex is even,  $d(3)=6$ , all others have degree 2]

#1 : Because every time an intermediate vertex is reached, say  $v$ ,  $d(v)$  is increased by two, one for going into  $v$  and other from moving out of  $v$ .

**Hamiltonian Path:** A path in a connected graph  $G$  that contains all the vertices of  $G$  is called a Hamiltonian path.

A Hamiltonian path is also known as a **Traceable path**.

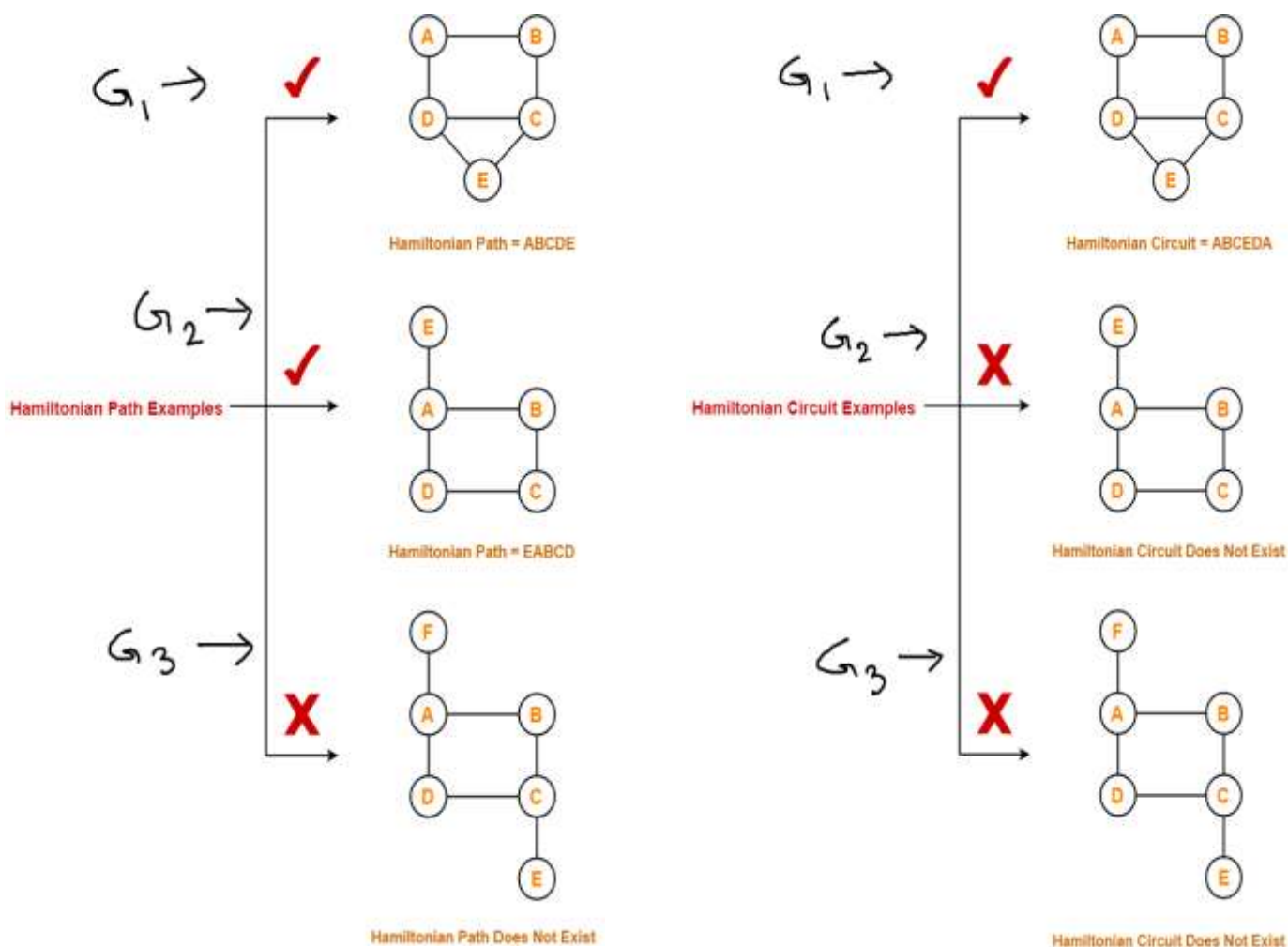
**Traceable Graph:** A graph  $G$  is traceable if it contains a Hamiltonian path.

**Hamiltonian Circuit:** If in a Hamiltonian path, the initial and terminal vertex coincide, it becomes a Hamiltonian circuit.

A Hamiltonian circuit is also known as a **Hamiltonian cycle**.

**Hamiltonian Graph:** A graph  $G$  containing a Hamiltonian cycle is known as a Hamiltonian graph.

**Examples:**



In the examples above,

$G_1$  is both a Traceable & Hamiltonian Graph

$G_2$  is a Traceable Graph but not a Hamiltonian Graph

$G_3$  is neither a Traceable Graph nor a Hamiltonian Graph

**Note 1:** A Hamiltonian cycle in a graph  $G$  contains all the vertices of  $G$  and is therefore a spanning cycle.

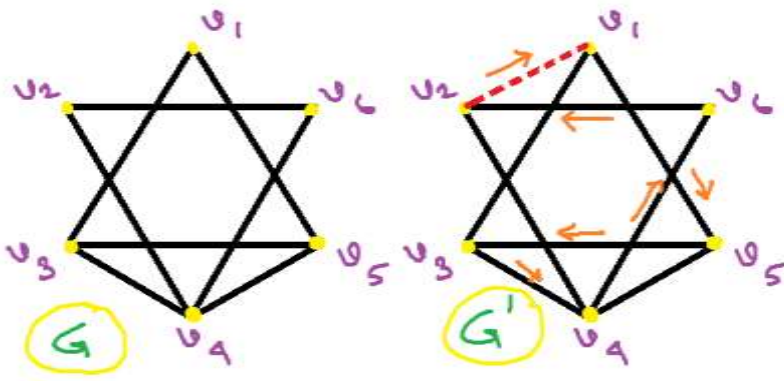
**Note 2:** If exactly one edge is removed from a Hamiltonian cycle, it becomes a Hamiltonian path. Thus, any Hamiltonian cycle with  $m$  edges will always contain a Hamiltonian path of length  $m - 1$ .

**Note 3:** Every Hamiltonian graph is connected.

**Note 4:** Every Hamiltonian graph contains a traceable graph; but the converse is not true.

**Note 5:** A circuit in a connected graph  $G$  is Hamiltonian if it includes every vertex of  $G$ . Hence a Hamiltonian circuit in a graph  $G$  with  $n$  vertices consists of exactly  $n$  edges.

**Maximal Non-Hamiltonian Graph:** Let  $G$  be a simple graph that is not Hamiltonian. If addition of a single edge connecting two non-adjacent vertices of  $G$ , makes the graph Hamiltonian, then  $G$  is called a maximal non-Hamiltonian graph.

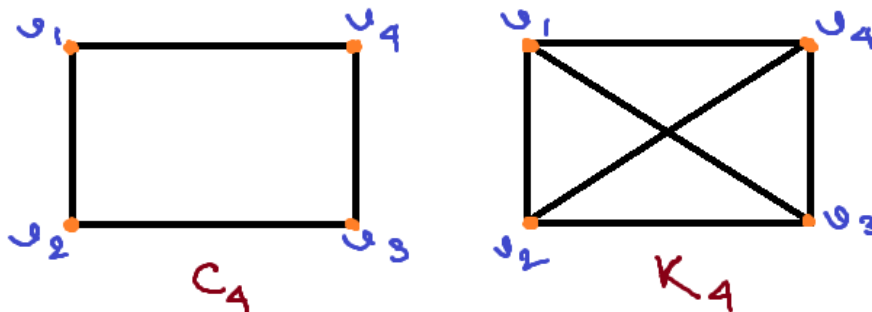


In the picture  $G$  is a Maximal non-Hamiltonian Graph, as by adding the edge  $v_1v_2$  we get a Hamiltonian circuit as,

$$v_3 \rightarrow v_4 \rightarrow v_6 \rightarrow v_2 \rightarrow v_1 \rightarrow v_5 \rightarrow v_3$$

### Further Examples:

1. Every cycle  $C_n (n \geq 3)$  is Hamiltonian as it contains the Hamiltonian circuit:  $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_n \rightarrow v_1$ .
2. Every complete graph  $K_n (n \geq 3)$  is Hamiltonian, as it contains the Hamiltonian circuit  $C_n$ .



**Note 6:** If a graph is Hamiltonian, then any supergraph with the same set of vertices containing it must also be Hamiltonian, since additional edges does not change the nature of a Hamiltonian graph.

**Result:** Let  $G$  be a simple graph with  $n(> 2)$  vertices, such that  $d(u) + d(v) \geq n$ , for each non-adjacent pair of vertices  $u, v \in V(G)$ , then  $G$  has a Hamiltonian circuit.

Let  $G$  be a non-Hamiltonian graph that satisfies the given condition.

From  $G$  we construct a maximal non-Hamiltonian graph  $G'$  by adding edges to  $G$  (adding one more edge to  $G'$  will make it Hamiltonian).

Since we have only added edges to  $G$  to obtain  $G'$ , the condition  $d(u) + d(v) \geq n$  is also true in  $G'$ .

Therefore  $G'$  must have a Hamiltonian path. Let us take that Hamiltonian path as  $u, x_1, x_2, \dots, x_{n-2}, v$ . (Adding the edge  $(u, v)$  will make  $G'$  a Hamiltonian circuit as  $u, x_1, x_2, \dots, x_{n-2}, v, u$ ).

If  $x_i$  is adjacent to  $u$  then  $x_{i-1}$  cannot be adjacent to  $v$  in  $G'$ . Because otherwise we will have a circuit like  $u, x_i, x_{i+1}, \dots, x_{n-2}, v, x_{i-1}, x_{i-2}, \dots, x_2, x_1, u$ , which is a Hamiltonian circuit, contradicting the fact that  $G'$  is non-Hamiltonian.

Now the number of vertices adjacent to  $u$  is  $d(u)$ . The above result effectively means that  $d(u)$  number of vertices cannot be adjacent to  $v$ .

Therefore,  $d(v) \leq (n - 1) - d(u)$  [ $\because$  maximum possible degree of  $v$  is obviously  $n - 1$ , and it has been just proved that  $d(u)$  number of vertices cannot be adjacent to  $v$ ]

Thus,  $d(u) + d(v) \leq n - 1$

But this is a contradiction, since according to the condition  $d(u) + d(v) \geq n$ .

Hence our initial assumption that  $G$  is non-Hamiltonian is wrong; i.e.  $G$  is Hamiltonian.

**Note:** This result is known as the Ore's Theorem. The condition mentioned in the theorem is only a sufficient condition but not a necessary condition for any graph to be Hamiltonian.

**Corollary:** Let  $G$  be a simple graph with  $n(> 2)$  vertices, such that  $d(v) \geq \frac{n}{2}, \forall v \in V(G)$ , then  $G$  has a Hamiltonian circuit.

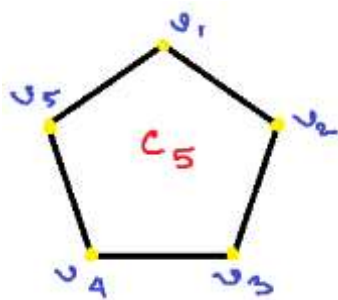
Let  $G$  be a graph with  $n(> 2)$  vertices and let the degree of every vertex in  $G$  be  $\geq \frac{n}{2}$ .

Then, the sum of degrees of any two non-adjacent vertices in  $G$  is  $\geq n$ .

Thus, by Ore's theorem,  $G$  is a Hamiltonian graph.

**Note:** This result is known as the Dirac's Theorem. The condition mentioned in the theorem is only a sufficient condition but not a necessary condition for any graph to be Hamiltonian.

### Example:



For the cycle graph  $C_5$ ,  $d(v_1) + d(v_3) = 2 + 2 = 4 < 5$  so it does not satisfy the condition of Ore's theorem.

Again,  $d(v_1) = 2 < \frac{5}{2} = 2.5$

and therefore it does not satisfy the condition of Dirac's theorem.

But  $C_5$  is Hamiltonian, as we have a Hamiltonian circuit:

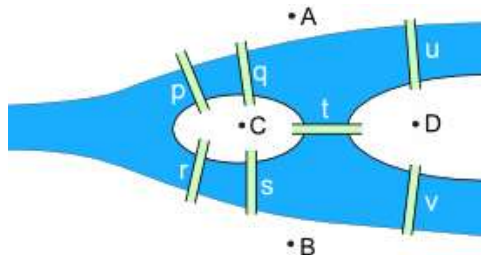
$$v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \rightarrow v_5 \rightarrow v_1$$

### Problems on Chapter 4:

1. Which of the following graphs has an Eulerian circuit and why?

- (A) Any  $k$ -regular graph where  $k$  is an even number.
- (B) A complete graph on 90 vertices
- (C) The complement of a cycle on 25 vertices
- (D) None of the above

2.

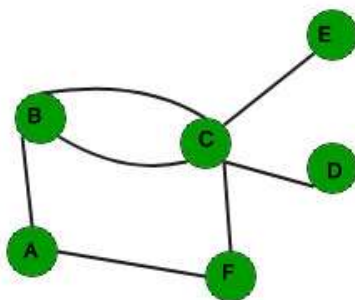


In the picture A,B,C,D are four land masses and p,q,r,s,t,u,v are seven bridges build across them.

Is it possible to visit each of A,B,C,D by crossing each of the seven bridges exactly once?

[Konigsberg bridge problem]

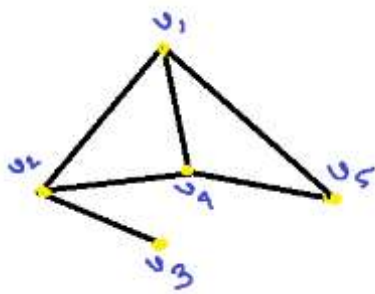
3.



Is the graph on the left

- (a) Eulerian Graph;
- (b) Semi-Eulerian Graph;
- (c) Hamiltonian Graph;
- (d) Traceable graph;
- (e) None of the above

4.



Examine if the graph on the left is edge-traceable or not.

5. Prove that a Hamiltonian graph cannot contain a cut vertex.
6. Prove that, no bipartite graph with odd number of vertices can be a Hamiltonian graph.
7. Show that a  $k$ -regular graph of order  $2k - 1$  is Hamiltonian.
8. Find  $m, n$  for which the complete bipartite graph  $K_{m,n}$  is Semi-Eulerian.
9. let  $G$  be a simple connected graph of order  $n (> 2)$  and size  $m$ . Then  $G$  is Hamiltonian if  $m \geq \frac{(n-1)(n-2)}{2} + 2$ .
10. Draw graphs of order 7, which are both Eulerian & Hamiltonian, with (a) Minimum number of edges, and (b) Maximum number of edges.

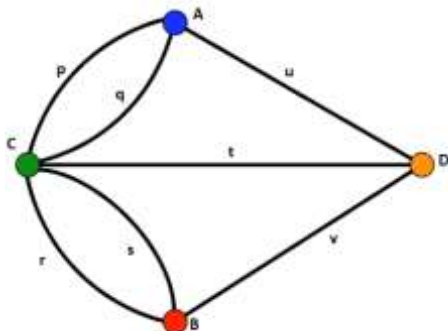
Hints & Answers:

1. Correct answer (C)

answer (A) is not correct as a  $k$ -regular graph may not be connected and therefore not Eulerian; answer (B) is not correct as  $d(v) = 89, \forall v \in V(G)$ , which is an odd number but we know that in a Eulerian graph degree of each vertex has to be even;

answer (C) is correct as all vertices in a cycle is of degree 2 and in its complement all vertex will have a degree of 22 [as in  $K_{25}, d(v) = 25 - 1 = 24 \Rightarrow d(v) = 24 - 2 = 22, \forall v \in V(\bar{G})$ ]; thus every vertex is of even degree and hence it is Eulerian.

2.



The picture on the left represents the given problem in a graph theoretic manner, in which each of the land masses are represented by the vertices A,B,C,D and the seven bridges are represented by the seven edges of this graph.

As is clear,  $d(A) = d(B) = d(D) = 3$  &  $d(C) = 5$ , all odd numbers, and thus not Semi-Eulerian or edge-traceable.

Hence it is not possible to visit all 4 destinations A,B,C,D by passing through all the bridges p,q,r,s,t,u,v exactly once.

3. Correct answer is (e);

Since the degree of more than two vertices are odd [ $d(d) = d(e) = 1; d(b) = 3$  &  $d(c) = 5$ ] it is neither (a) nor (b);

Also as there are two vertices of degree 1 and a cut vertex (c), there can be no Hamiltonian path nor Hamiltonian cycle, hence neither (c) nor (d) is correct.

4. A graph is edge-traceable if it is either an Euler graph or a semi-Euler graph.

However in this graph,  $d(v_1) = d(v_2) = d(v_4) = 3$  &  $d(v_3) = 1$  [all odd degrees]

Hence it is not edge-traceable.

5. Let  $G$  be a Hamiltonian graph.

Let us also assume that  $G$  contain a cut vertex  $v$ . Then  $G - v$  is disconnected. Let  $x, y$  be two vertices from two different components in  $G - v$ .

Now  $G$  contains a Hamiltonian cycle. If we start the cycle at  $v$ , without loss of generality, assume that it then passes through  $x$  and then  $y$ , and since the cycle is Hamiltonian, the path between  $x$  and  $y$  won't pass through  $v$ . [ as in a Hamiltonian graph, every vertex appears only once]

Then if we remove  $v$ , it won't affect the path between  $x$  and  $y$ , which means that  $x, y$  belong to the same component of  $G - v$ , contradicting the choice of  $x$  and  $y$ .

Thus, there can be no cut vertex in  $G$ . Hence the result.

6. Let  $G = (V, E)$  be a bipartite graph with bipartition  $V = V_1 \cup V_2$ . Then, every Hamiltonian cycle in  $G$  must visit all the vertices of  $G$ .

As  $G$  is bipartite, this cycle will have the form  $a_1 \rightarrow b_1 \rightarrow a_2 \rightarrow b_2 \rightarrow \dots \rightarrow a_k \rightarrow b_k \rightarrow a_1$ , where  $a_i \in V_1$  &  $b_i \in V_2, 1 \leq i \leq k$ .

As it is a cycle, it begins and ends at the same vertex,  $a_1$ .

Thus, we see that the number of vertices in the graph must be  $2k$ .

7. Graph  $G$  is  $k$ -regular with order  $n = 2k - 1$ . Then  $d(v) = k, \forall v \in V(G)$

Hence,  $d(v) - \frac{n}{2} = k - \frac{2k-1}{2} = \frac{1}{2} > 0 \Rightarrow d(v) > \frac{n}{2}, \forall v \in V$

Hence, by Dirac's theorem  $G$  is Hamiltonian.

8. A graph is Semi-Eulerian if either all its vertices are even or two of its vertices are odd

Case1: If both  $m, n$  are even,  $K_{m,n}$  is Eulerian and hence semi-Eulerian.

Case2: If  $m$  is odd and  $n = 2$ , then only two vertices belonging to the second group of vertices  $V_2$  will be odd, and hence the graph is semi-Eulerian.

Case3: If  $m = 2$  and  $n$  is even, then only two vertices in  $V_1$  will have odd degree, and hence the graph is semi-Eulerian.

Case4: If  $m = n = 1$ , then also two vertices will have odd degree and hence  $K_{1,1}$  is semi-Eulerian.

9. Let, the condition holds.

If  $G = K_n$ , then we are done as  $K_n$  is Hamiltonian.

So, let us assume that  $G \neq K_n$ ; then  $\exists u, v \in G$  such that they are non-adjacent.

Let us construct  $G'$  from  $G$  by deleting the vertices  $u, v$ .

Then,  $m = |E(G)| = |E(G')| + d(u) + d(v)$

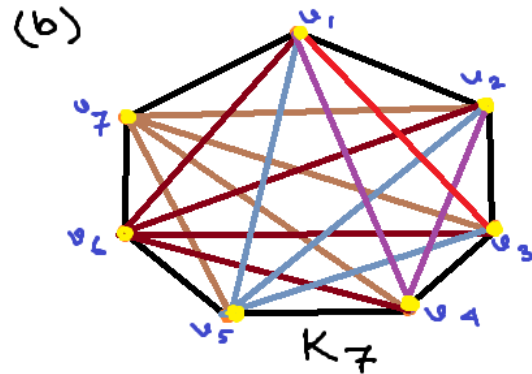
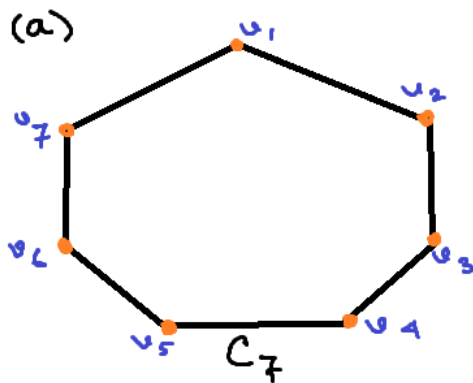
But  $E(G')$  is a simple graph of order  $n - 2 \Rightarrow |E(G')| \leq |E(K_{n-2})| = \frac{(n-2)(n-3)}{2}$

Hence,  $\frac{(n-1)(n-2)}{2} + 2 \leq m = |E(G)| = |E(G')| + d(u) + d(v) \leq \frac{(n-2)(n-3)}{2} + d(u) + d(v)$

Simplifying which gives,  $d(u) + d(v) \geq n$ .

Hence using Ore's theorem, we can say that  $G$  is Hamiltonian.

10.



(a) Any cycle with 7 vertices must have 7 edges.

In  $C_7$ , there is a Hamiltonian cycle as  $v_1 \rightarrow \dots \rightarrow v_7 \rightarrow v_1$ , hence Hamiltonian.  
also in  $C_7, d(v) = 2, \forall v$ , which is even, hence Eulerian.

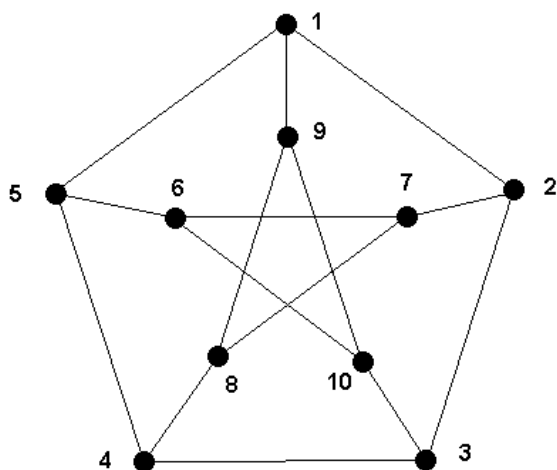
(b) Again in  $K_7, d(v) = 6, \forall v$ , which is even, hence Eulerian.

also  $K_7$  contains the Hamiltonian cycle  $C_7$ , and hence Hamiltonian.

also  $K_7$  is the graph of largest size with 7 vertices.

Assignments on Chapter 4:

1.  $G$  is a simple graph. Some vertices of  $G$  are of odd degree. Add a node  $v$  to  $G$  and make it adjacent to each odd degree vertex of  $G$ . Show that the resultant graph will be an Euler graph.
2. Prove that a complete Bipartite graph  $K_{m,n}$  has a Hamilton cycle iff  $m = n, \forall m, n \geq 2$ .
3. Let  $G$  be a regular graph of even order and odd size. Show that  $G$  cannot be Eulerian.
4. Show that a simple connected graph  $G$  of order 4 and size 5 is Hamiltonian.
5. Show that the Petersen's graph has a Hamiltonian path but not a Hamiltonian cycle.



6. Examine if the graph  $G$  in the following figure is (a) Eulerian ; (b) Hamiltonian.

