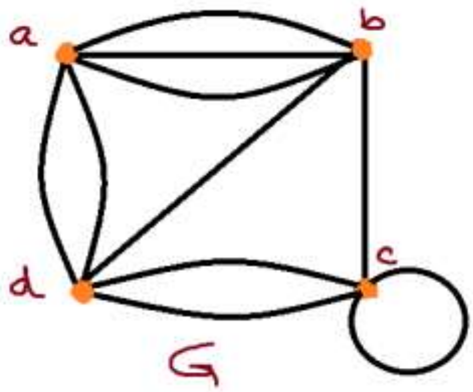
Adjacency Matrix:

Let, $G(V, E)$ be a graph of order n, with $V(G) = \{v_1, \dots, v_n\}$. Then the adjacency matrix $A(G)$ is a square matrix of order n, where ij-th element a_{ij} is the number of edges between v_i and v_j.

i.e. $A(G) = (a_{ij})_{n\times n}$, where, $a_{ij} = \begin{cases} number\ of\ edges\ between\ v_i, v_j, i \neq j \\ number\ of\ loops\ at\ v_i\ , i = j \end{cases}$

clearly for an undirected graph, an edge between v_i & v_j is the same as an edge between v_j & v_i and hence an adjacency matrix is symmetric about the principal diagonal.

Example:



The adjacency matrix for the graph is as follows:

$A(G) =$

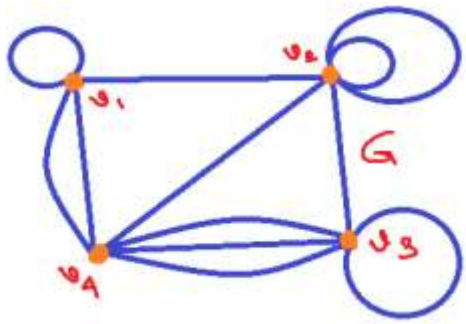
$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{pmatrix} \end{array}$$

As there are three edges between $a, b \Rightarrow a(1,2) = 3$
As there is only one edge between $b, c \Rightarrow a(2,3) = 1$
Also there is a loop at $c \Rightarrow a(3,3) = 1$
and so on…

Conversely, given any square matrix A of order n, we can always construct a graph G with n vertices such that A is the adjacency matrix of G.

Example:

Given, $A = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 2 & 1 & 3 & 0 \end{pmatrix}$, we can draw the graph G:



Note 1: Different adjacency matrices are possible for different ordering of the vertices. But if the graph is a complete graph, then, $a_{i,j} = 1, \forall i \neq j$, and hence in this case the adjacency matrix is same for all possible ordering of the vertices.

For a simple graph G, adjacency matrix is given as,

$$A(G) = (a_{i,j})_{n\times n}\ ;\ a_{i,j} = \begin{cases} 1, \exists\ an\ edge\ between\ v_i, v_j \\ 0, \nexists\ an\ edge\ between\ them \end{cases}$$

Note 2: For a simple graph G, the degree of any vertex is given by the Row-Sum (Column-Sum) of the corresponding row (column) of the adjacency matrix. The Row-Minima (Column-Minima)gives $\delta(G)$ and the Row-Maxima (Column-Maxima) gives $\Delta(G)$.

Powers of Adjacency matrix:

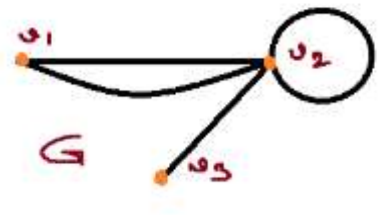
Let, G be a given simple graph of order n and $A(G)$ be its adjacency matrix. Let $b_{i,j}$ be the i,j-th entry $(i \neq j)$ in the matrix $[A(G)]^2 = B$(say).

Then, $b_{i,j}$ = number of 1's in the dot product of i-th row and j-th column (j-th row) of $A(G)$
= number of positions in which both i-th and j-th rows of $A(G)$ have 1's
= number of vertices that are adjacent to both i-th and j-th vertices
= number of edge sequences of length two between i-th and j-th vertices

Similarly, the i,i-th diagonal entry in $[A(G)]^2$ is the number of 1's in the i-th row (or column) of $A(G)$. Thus, the value of each diagonal entry in $[A(G)]^2$ equals the degree of the corresponding vertex (as there is no self-loop).

Note 3: This result only holds for a simple graph.

Example:



In G the adjacency matrix w.r.t. the vertex set $\{v_1, v_2, v_3\}$ is

$$A(G) = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow [A(G)]^2 = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 6 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

According to principal diagonal, $d(v_1) = 4; d(v_2) = 6; d(v_3) = 1$

But,$d(v_1) = 2; d(v_2) = 4; d(v_3) = 1$, hence the result fails in this case, as the graph is not simple.

Again, a matrix commutes with any matrix that are its own power. Hence

$$A(G).[A(G)]^2 = [A(G)]^2.A(G) = [A(G)]^3$$

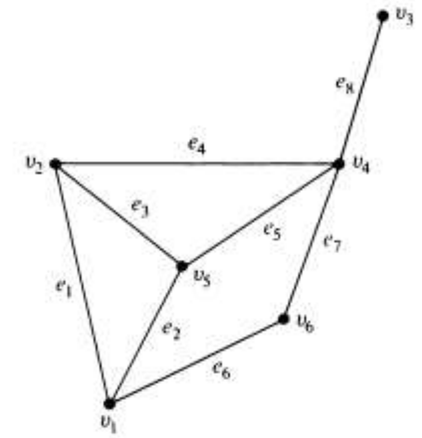
Also, since the product of two symmetric matrices that commute is again a symmetric matrix, then $[A(G)]^3$ is asymmetric.

ij-th entry in $[A(G)]^3$ = dot product of i-th row of $[A(G)]^2$ and j-th column (or row) of $A(G)$
$= \sum_{k=1}^{n} ik$-th entry of $[A(G)]^2.\, kj$-th entry of $A(G)$
$= \sum_{k=1}^{n}$ number of different edge sequences of 3 edges from i-th to j-th vertex via k-th vertex
= number of different edge sequences of length 3 between i-th and j-th vertices

This result can be further generalized as:

Let, $A(G)$ be the adjacency matrix of a simple graph G. Then the ij-th entry in $[A(G)]^r$ is the number of different edge sequences of r edges between vertices v_i and v_j.

Example:



$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

for the graph above the adjacency matrix is denoted by X.

Then we have,

$$X^2 = \begin{bmatrix} 3 & 1 & 0 & 3 & 1 & 0 \\ 1 & 3 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 1 & 2 & 1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 0 & 2 & 2 \end{bmatrix}.$$

Clearly in this case the diagonal elements of X^2 give the degrees of the vertices, as:

$$d(v_1) = 3; d(v_2) = 3; d(v_3) = 1; d(v_4) = 4;$$
$$d(v_5) = 3; d(v_6) = 2$$

Also, we have,

$$X^3 = \begin{bmatrix} 2 & 7 & 3 & 2 & 7 & 6 \\ 7 & 4 & 1 & 8 & 5 & 2 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 2 & 8 & 4 & 2 & 8 & 7 \\ 7 & 5 & 1 & 8 & 4 & 2 \\ 6 & 2 & 0 & 7 & 2 & 0 \end{bmatrix}.$$

Let us for example consider the 1,5-th element of X^3, i.e. 7
this means there are seven edge sequences of length 3 between v_1 and v_5; these are

$$\{e_1, e_1, e_2\}, \{e_2, e_2, e_2\}, \{e_6, e_6, e_2\}, \{e_2, e_3, e_3\},$$
$$\{e_6, e_7, e_5\}, \{e_2, e_5, e_5\}, \{e_1, e_4, e_5\}$$

Clearly, out of this seven, only two, i.e. $\{e_6, e_7, e_5\}$ and $\{e_1, e_4, e_5\}$ are paths.

Similarly, we can construct X^4, X^5, … and find edge sequences of lengths 4,5,… and so on.

Corollary:

An important corollary to this result is as follows:

If X is the adjacency matrix of a graph G with n vertices, and $Y = X + X^2 + X^3 + \cdots + X^{n-1}$, then G is disconnected if and only if there exists at least one non-diagonal entry in Y that is zero, i.e. if $Y = (y_{i,j})$, then $y_{i,j} = 0$, for some $i, j, i \neq j$.

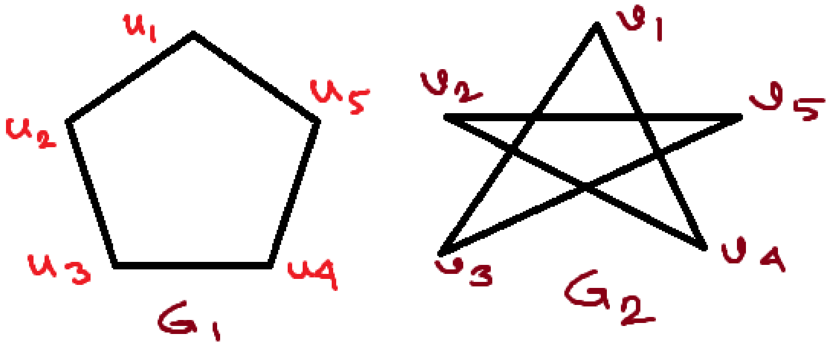
If a non-diagonal entry in Y is zero, say ij-th entry, then it means that the ij-th entry is zero in $X, X^2, X^3, \ldots, X^{n-1}$

This means that there is no path of length $1, 2, \ldots, n-1$ between v_i and v_j. Since there are n vertices, which in turn means that there is no path between v_i and v_j at all. Hence the graph is disconnected and conversely.

Note 4:

Let, G_1, G_2 be two graphs of same order. Then $G_1 \cong G_2$, if $A(G_1) = A(G_2)$ holds for some ordering of their vertex sets.

Example:



Adjacency matrix of G_1 with the vertex set $\{u_1, u_2, u_3, u_4, u_5\}$ is

$$A(G_1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Again, Adjacency matrix of G_2 with the vertex set $\{v_1, v_3, v_5, v_2, v_4\}$ is

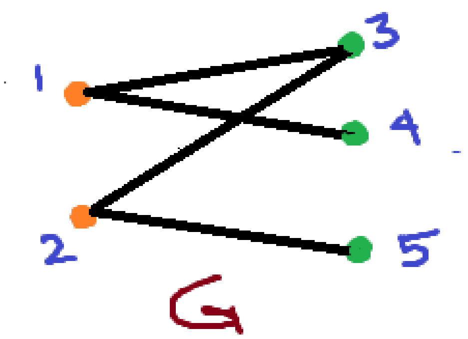
$$A(G_2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Since, $A(G_1) = A(G_2) \Rightarrow G_1 \cong G_2$.

Note 5:

If $G = (V_1, V_2, E)$ be a bi-partite graph with $|V_1| = n_1$ & $|V_2| = n_2$, then the adjacency matrix $A(G)$ can be expressed as, $\begin{pmatrix} O & P \\ P^T & O \end{pmatrix}$, for some ordering of vertex set $V_1 \cup V_2$, where P is an $n_1 \times n_2 (0,1)$ matrix[#1] and O is a null matrix.

Example:



In the graph G above, we have $V_1 = \{1,2\}$ & $V_2 = \{3,4,5\}$

The adjacency matrix for the vertex set $\{1,2,3,4,5\}$ is given as

$$A(G) = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} O_2 & P \\ P^T & O_3 \end{pmatrix}$$

Where, $P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ is a $(0,1)$ matrix of order 2×3.

Incidence Matrix:

Let G be a graph of order n and size m. Let $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_m\}$ be the vertex and edge set respectively. Then the incidence matrix, denoted by $I(G)$, is $n \times m$ matrix defined as

$$I(G) = (a_{ij})_{n\times m} \text{ ; where } a_{i,j} = \begin{cases} 0, & if\ e_j\ is\ not\ incident\ on\ v_i \\ 1, & if\ e_j\ is\ incident\ on\ v_i\ and\ is\ not\ a\ loop \\ 2, & if\ e_j\ is\ a\ loop\ at\ the\ vertex\ v_i \end{cases}$$

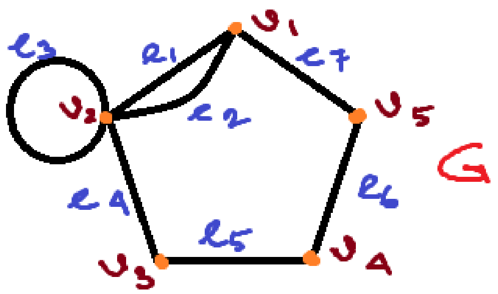
In case of a simple graph G, it is defined as

$$I(G) = (a_{ij})_{n\times m} \text{ ; where } a_{i,j} = \begin{cases} 1, & if\ e_j\ is\ incident\ on\ v_i \\ 0, & otherwise \end{cases}$$

#1 A $(0,1)$ matrix is also known as a Boolean matrix

Thus, for a simple graph the incidence matrix is a Boolean matrix [(0,1) matrix].

Example:



Here, $V(G) = \{v_1, v_2, \dots, v_5\}$

And, $E(G) = \{e_1, e_2, \dots, e_7\}$

With respect to these vertex and edge sets

The incidence matrix is given below:

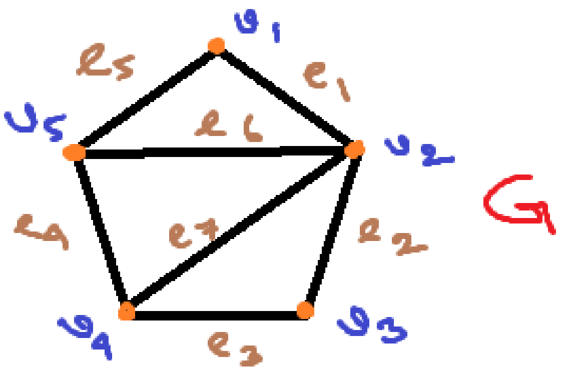
$I(G) =$

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Similarly given any incidence matrix, we can draw the corresponding graph.

Example:

Given, $I(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$, the corresponding graph is given below.



Note 1:

Different incidence matrices may be generated for different ordering of the vertex and edge set.

Note 2:

Let G be a graph of order n and degree m. Then the sum of the elements in i-th row of an incidence matrix $I(G)$ is equal to $d(v_i), \forall i = 1,2,\dots,n$.

Corollary: If row-sum is zero for j-th row then v_j is isolated, i.e. the graph G is disconnected.

Note 3:

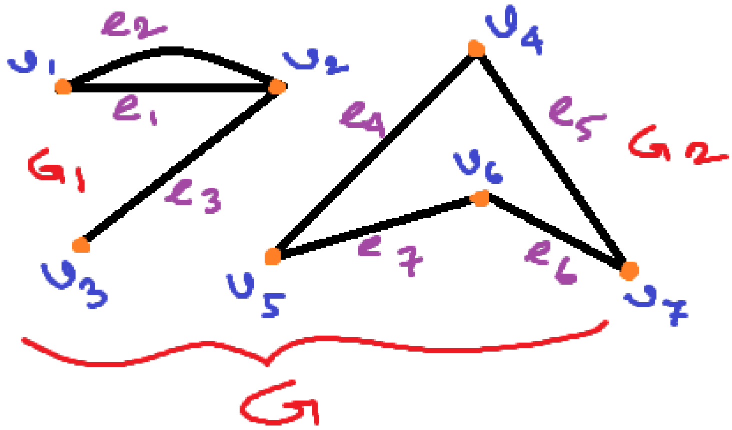
Two columns of the incidence matrix are identical, if there exists a pair of parallel edges in G.

Note 4:

If G is a disconnected graph with components G_1 and G_2, the incidence matrix can be expressed as $I(G) = \begin{pmatrix} I(G_1) & O \\ O^T & I(G_2) \end{pmatrix}$, where $I(G_1)$ and $I(G_2)$ are incidence matrices of components G_1 and G_2 respectively, and O is a null matrix.

This results from the fact that no edge in G_1 is incident on any vertex of G_2 and vice-versa.

Example:



In the graph G,

$$V(G_1) = \{v_1, v_2, v_3\},$$
$$V(G_2) = \{v_4, v_5, v_6, v_7\},$$
$$E(G_1) = \{e_1, e_2, e_3\},$$
$$E(G_2) = \{e_4, e_5, e_6, e_7\}$$

With respect to the natural ordering of vertices and edges, the incidence matrix is as follows:

$$I(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} I(G_1) & O_{3\times 4} \\ O^T & I(G_2) \end{pmatrix}$$

Where, $I(G_1) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is the incidence matrix of G_1;

$I(G_2) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ is the incidence matrix of G_2 ; and $O_{3\times 4}$ is a null matrix of order 3×4.

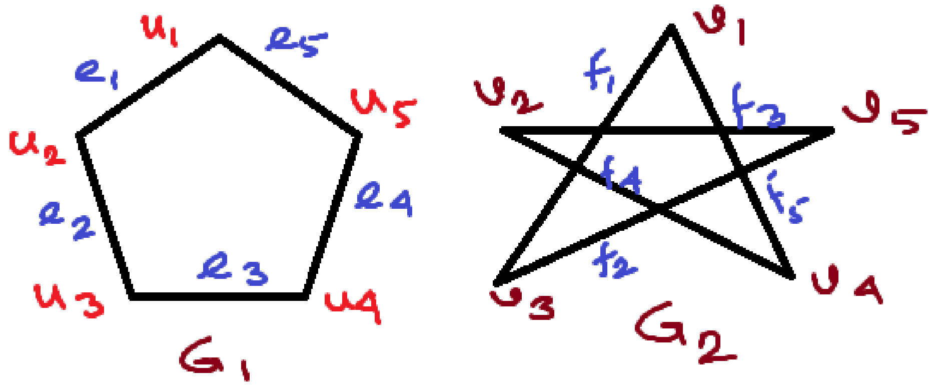
Note 5:

Let G_1, G_2 be two graphs of same order. Then $G_1 \cong G_2$, if and only if $I(G_1) = I(G_2)$ holds for some ordering of the vertex and edge sets of G_1, G_2.

Let the graphs $G_1$and G_2 be isomorphic. Then there is a one-one correspondence between the vertices and edges in G_1 and G_2 such that the incidence relation is preserved. Thus $I(G_1)$ and $I(G_2)$ are either same or differ only by permutation of rows and columns.

The converse follows, since permutation of any two rows or columns in an incidence matrix simply corresponds to relabelling the vertices and edges of the same graph.

Example:



Taking the vertices as $V(G_1) = \{u_1, u_2, u_3, u_4, u_5\}$ and edges as $E(G_1) = \{e_1, e_2, e_3, e_4, e_5\}$

We have, $I(G_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$;

Again, taking vertices as $V(G_2) = \{v_1, v_3, v_5, v_2, v_4\}$ and edges as $E(G_2) = \{f_1, f_2, f_3, f_4, f_5\}$

we have $I(G_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$

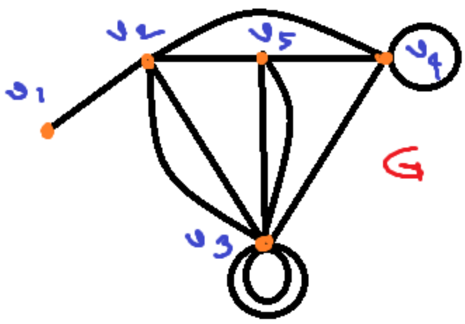
clearly therefore, $I(G_1) = I(G_2) \Rightarrow G_1 \cong G_2$

Note 6:

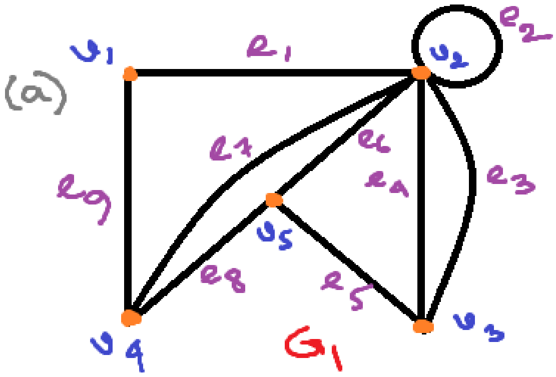
There are exactly two 1's in any column of an incidence matrix $I(G)$.

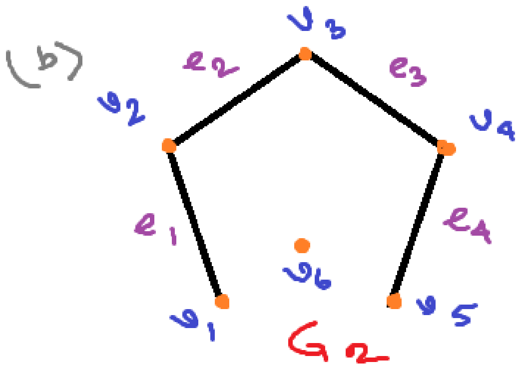
Problems on Chapter 5:

1. Find the adjacency matrices of the following graph with respect to the ordering
   (a) $\{v_1, v_3, v_2, v_4, v_5\}$; (b) $\{v_1, v_2, v_5, v_4, v_3\}$;



2. Find the adjacency matrix of the complete bi-partite graph $K_{2,3}$.
3. Construct the incidence matrix of the following graphs with respect to the natural ordering of vertex and edge sets:





4. Find the adjacency as well as the incidence matrix for the graph K_6.

Hints & Answers:

1. (a) $A(G) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 2 \\ 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 0 \end{pmatrix}$; (b) $A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 1 & 2 \end{pmatrix}$;
2. For $K_{2,3}$, let $V_1 = \{v_1, v_2\}$ and $V_2 = \{v_3, v_4, v_5\}$, such that $V = V_1 \cup V_2$
   Then, with respect to the natural order of vertices, the Adjacency matrix is given by:
   $$A(G) = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} O & P \\ P^T & O \end{pmatrix}$$
   Note: Being a complete bipartite graph, it does not matter if the order within V_1, V_2 are altered.

3. (a) $I(G)=\begin{pmatrix}1&0&0&0&0&0&0&0&1\\1&2&1&1&0&1&1&0&0\\0&0&1&1&1&0&0&0&0\\0&0&0&0&0&0&1&1&1\\0&0&0&0&1&1&0&1&0\end{pmatrix}$; (b) $I(G)=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{pmatrix}$

4. Complete graph K_6 has 6 nodes and 15 edges.

   Hence, $A(G)=\begin{pmatrix}0&1&1&1&1&1\\1&0&1&1&1&1\\1&1&0&1&1&1\\1&1&1&0&1&1\\1&1&1&1&0&1\\1&1&1&1&1&0\end{pmatrix}$;

$$I(G)=\begin{pmatrix}1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\1&0&0&0&0&1&1&1&1&0&0&0&0&0&0\\0&1&0&0&0&1&0&0&0&1&1&1&0&0&0\\0&0&1&0&0&0&1&0&0&1&0&0&1&1&0\\0&0&0&1&0&0&0&1&0&0&1&0&1&0&1\\0&0&0&0&1&0&0&0&1&0&0&1&0&1&1\end{pmatrix}$$

Assignments on Chapter 5:

1. Let, G be a graph with adjacency matrix $A(G)=\begin{pmatrix}0&1&1&2\\1&1&1&2\\1&1&2&0\\2&2&0&0\end{pmatrix}$. Without drawing the graph G examine if it is connected or not.
2. Let, G be a graph with adjacency matrix $A(G)=\begin{pmatrix}1&2&0\\2&0&1\\0&1&0\end{pmatrix}$, for the vertex set $V(G)=\{v_1,v_2,v_3\}$. Find the number of distinct walks of length 3 from (a) v_1 to v_3; (b) v_1 to v_2 & (c) v_2 to v_3. Verify the result by actually drawing the graph G.
3. Draw the graph G with an incidence matrix $I(G)=\begin{pmatrix}1&0&0&0&1&1&2\\1&1&0&0&0&0&0\\0&1&1&1&0&0&0\\0&0&1&1&1&1&0\end{pmatrix}$.
4. Show that the following graphs are isomorphic.