U.G. 5th Semester Examination - 2020
MATHEMATICS
[HONOURS]

Course Code : MATH-H-CC-T-11

(Partial Differential Equations & Applications)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The notations and symbols have their usual meanings.

1. Answer any ten questions. $2\times 10=20$

a. What do you mean by complete integral and general integral of a first order PDE?

b. Eliminate the arbitrary function and form the PDE from $z = xy + f(x^2 + y^2)$.

c. Interpret the equation $Pp + Qq = R$ and $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ geometrically and establish the relationship between the two.

d. Eliminate the constants a and b from $ax^2 + by^2 + z^2 = 1$ and obtain the PDE.

e. Find a complete integral of $p^2z^2 + q^2 = 1$.

f. Classify the PDE $z_{xx} - xz_{xy} + 4z_{yy} + 4z_y = 0$.

g. Give examples of a linear, semilinear, quasilinear and nonlinear first order PDE.

h. Find the characteristics of $\sin^2 x \; z_{xx} + 2\cos x \, z_{xy} - z_{yy} = 0$.

i. If $z = f(x^2 - y) + g(x^2 + y)$, where f and g are arbitrary, prove that

$$z_{xx} - \frac{1}{x} z_x = 4x^2 z_{yy} .$$

j. Find the PDE describing the set of all right circular cones whose axis coincides with z axis.

k. Explain the order and degree of a PDE with examples.

l. Prove that the direction cosines of the tangent at the point (x, y, z) to the conic $lx^2 + my^2 + nz^2 = 1$, $x + y + z = 1$ are proportional to $(my - nz, nz - lx, lx - my)$.

m. Show that there always exists an integrating factor for a Pfaffian differential equation in two variables.

n. State Cauchy-Kowalewski theorem.

o. Determine the region where the given PDE $x\, u_{xx} + u_{yy} = x^2$ is hyperbolic, parabolic or elliptic.

[Turn over]

2. Answer any four questions. $5\times 4=20$

a. Find the equation of the integral surface of the differential equation $2y(z-3)p+(2x-z)q=y(2x-3)$ which passes through the circle $z=0, x^2+y^2=2x$.

b. Find the surface which intersects the surfaces of the system $z(x+y)=c(3z+1)$ orthogonally and which passes through the circle $x^2+y^2=1,\ z=1$.

c. Reduce the equation $(n-1)^2\, z_{xx}-y^{2n}\, z_{yy}=ny^{2n-1}z_y$ to a canonical form, where n is an integer and find its general solution.

d. i) Solve the equation $(D^2-D'^2+2D-2D')u=e^{3x-2y}$.

ii) If the operator $F(D,D')$ is reducible, then show that the order in which the linear factor occur is meaningless.

e. Find the orthogonal trajectories on the cone $x^2+y^2=z^2\, tan^2\, \alpha$ of its intersections, with the family of planes parallel to $z=0$.

f. Solve $p+3q=5z+\tan(y-3x)$.

3. Answer any two questions. $10\times 2=20$

a. i. Solve the problem by method of characteristic $pz+q=1$ with initial data $y=x, z=\frac{x}{2}$.

ii. Find a complete integral of the equation $p^2x+q^2y=z$. 5+5

b. i. What do mean by compatibility of two first order PDEs ? Derive a necessary and sufficient condition for the compatibility of the two first order PDEs $f(x,y,z,p,q)=0$ and $g(x,y,z,p,q)=0$.

ii. Show that the equations $p^2+q^2=1$ and $(p^2+q^2)x=pz$ are compatible and find their common solutions. (1+5)+4

c. Solve the one dimensional problem of transverse vibration of a finite string
$u_{tt} = c^2\, u_{xx}\,, 0 < x < l, t > 0$ with $u(x,0) = f(x), u_t(x,0) = g(x), 0 \le x \le l$ and $u(0,t) = u(l,t) = 0\,, t > 0$ using method of separation of variables.

Further solve the problem $u_{tt} = c^2\, u_{xx}, 0 < x < 1, t > 0$ with $u(0,t) = u(1,t) = 0, t > 0$ and $u(x,0) = x(1-x), u_t(x,0) = 0\,, 0 \le x \le 1$. 7+3

d. i. Solve the heat conduction problem for a finite rod of length l, $\frac{\partial T}{\partial t} - k\frac{\partial^2 T}{\partial x^2} = 0, 0 \le x \le l,\ t \ge 0$ subject to the boundary conditions $T(0,t) = T(l,t) = 0,\ t \ge 0$ and the initial condition $T(x,0) = f(x),\ 0 \le x \le l$.
ii. Prove that the solution is unique.
iii. Solve $T_t = T_{xx}, 0 < x < l,\ t > 0$, with $T(0,t) = T(l,t) = 0$ and $T(x,0) = x(l-x)$.
4+3+3.