U.G. 3rd Semester Examination - 2020
MATHEMATICS
[HONOURS]
Course Code : MATH-H-CC-T-06

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Let G be the set of all real 2×2 matrices $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ where $a \neq 0$. Prove that G is a group under matrix multiplication.

b) Is the dihedral group D_3 Abelian?

c) Give an example of group elements a and b for a suitable group G with the property that $a^{-1}ba^{-1} \neq b$.

d) Suppose G is a group with the property that if $ab = ca$ then $b = c$ for all $a, b, c \in G$. Then prove that G is Abelian.

e) Let G be the set of all real 2×2 matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ where $ad \neq 0$. Find a subgroup of G of order 4.

f) Let $\mathbb{Q}$ be the group of rational numbers under addition and $\mathbb{Q}^*$ be the group of non zero rational numbers under multiplication. In $\mathbb{Q}$, list the elements in $\langle \frac{1}{2} \rangle$. In $\mathbb{Q}^*$, list the elements in $\langle \frac{1}{2} \rangle$.

g) Show that $U(14)$ is cyclic.

h) Give an example to show that union of two subgroups of a group may not be a subgroup.

i) Prove that the mapping from $U(16)$ to itself given by $x \to x^3$ is an automorphism.

j) Prove that $U(72)$ is isomorphic to the direct product of $U(9)$ and $\mathbb{Z}_5$.

k) Let G be a group and let $a \in G$. Prove the mapping $\phi_a : G \to G$ defined by $\phi_a(x) = a^{-1}xa$ is an automorphism of G.

l) How many elements of order 9 does $\mathbb{Z}_3 \oplus \mathbb{Z}_9$ have?

m) Let the order of a group G be 33. Then show that G must have an element of order 3.

n) Let G be a group and H be a subgroup of G. Let $a \in G$. Prove that $\alpha H = H$ if and only if $a \in H$.

o) Let G be a nontrivial finite group. Show that G has an element of prime order.

2. Answer any **four** questions. 5×4 =20

i) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be a member of the group $SL(2, \mathbb{Z}_{43})$. Then find the order of A? 5

ii) a) Determine the number of cyclic subgroups of order 15 in $\mathbb{Z}_{90} \oplus \mathbb{Z}_{36}$.

b) How many cyclic subgroups of D_4 have? List them. Find a non cyclic subgroup of D_4 of order 4. 3+2

iii) a) Let G be a cyclic group, which has exactly three subgroups: G itself, trivial subgroup, and a subgroup of order 7. Then what is the group G?

b) Are $U(10)$ and $U(12)$ isomorphic? Justify your answer. 3+2

iv) Compute $\mathrm{Aut}(\mathbb{Z}_{10})$. 5

v) a) Prove that $SL(2, \mathbb{R})$ is a normal subgroup of $GL(2, \mathbb{R})$.

b) Determine the subgroup lattice for $U(12)$. 3+2

vi) a) Find the order of element $5 + \langle 6 \rangle$ in the factor group $\mathbb{Z}_{18}/\langle 6 \rangle$?

b) Suppose k is a divisor of n. Prove that $\mathbb{Z}_n/\langle k \rangle$ is isomorphic with $\mathbb{Z}_k$. 2+3

3. Answer any **two** questions: 10×2=20

i) a) Let G be an Abelian group and let $H = \{g \in G : o(g) \text{ divides } 12\}$. Then prove that H is a subgroup of G. Is it possible to replace 12 by some other positive integer?

b) Prove that if G is a group with the property that the square of every element is the identity, then G is Abelian. 5+5

ii) a) Let G be a group and a be an element of G of order n. Prove that for each integer k between 1 and n, $o(a^k) = o(a^{n-k})$.

b) How many generators do the group $\mathbb{Z}_{2021}$ have? 5+5

iii) Let $G = \{a + ib : a, b \in \mathbb{Q}\}$

and $H = \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} : a, b \in \mathbb{Q} \right\}$

Then prove that G and H are isomorphic under addition. Observing that G and H are closed under multiplication, does your isomorphism preserve multiplication also? Justify. 5+5

iv) a) Find the largest order of any element in $U(900)$?

b) Let $\{3^m 6^n : m, n \in \mathbb{Z}\}$ under multiplication. Prove that G is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

c) Suppose g and h induce the same inner automorphism of a group G. Prove that $gh^{-1} \in Z(G)$. 4+3+3

v) a) Let for any integer greater than 1, $\phi(n)$ denotes the number of integers less than *n* and relatively prime to *n*. Then prove that if *a* is any integer relatively prime to *n*, then $a^{\phi(n)} = 1$ modulo *n*.

b) Prove that there is no isomorphism between the group of rational numbers under addition and the group of non zero rational numbers under multiplication.

5+5

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