2021

MATHEMATICS

[HONOURS]

Paper : VI

Full Marks : 100 Time : 4 Hours

The figures in the right-hand margin indicate marks.

Symbols and Notations have their usual meaning.

1. Answer any **five** questions: 1×5=5

 a) Show that $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$ does not exist.

 b) Examine the convergence of $\int_0^1 \frac{dx}{\sqrt{1-x}}$.

 c) If $L[F(t)] = f(p)$ then prove that for $\lambda > 0$, $L[F(\lambda t)] = \frac{1}{\lambda} f\left(\frac{p}{\lambda}\right)$.

 d) Show that $z = f(x^2 y)$, where f is differentiable satisfying $x\left(\frac{\partial z}{\partial x}\right) = 2y\left(\frac{\partial z}{\partial y}\right)$.

 e) Show that the function f defined as

 $$f(x) = \frac{1}{2^n}, \text{ when } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}, (n = 0,1,2,...)$$
 $$= 0 \text{ when } x = 0,$$

 is integrable on [0, 1].

 f) Find the radius of convergence of the series $x + \frac{x^2}{2^2} + \frac{2!}{3^3}x^3 + \frac{3!}{4^4}x^4 + ...$

 g) Show that the sequence $\{f_n\}$, where $f_n(x) = \tan^{-1} nx, x \ge 0$ is uniformly convergent in any interval [a, b], a>0, but is only point-wise convergent is [0, b].

2. Answer any **ten** questions: 2×10=20

 a) If f is bounded and integrable on $[-\pi, \pi]$ and if a_n, b_n are its Fourier coefficients then prove that $\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ converges.

 b) Show that $f(xy, z-2x) = 0$ satisfies, under suitable conditions, the equation $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2x$. Find those conditions.

 c) Find the stationary points of the function $f(x,y,z) = (x+y+z)^3 - 3(x+y+z) - 24xyz + 27$.

 d) Evaluate $\iint_R [x+y]\,dx\,dy$, over the rectangle $R = [0,1;0,2]$, where [x+y] denotes the greatest integer less than or equal to (x+y).

 e) Locate and classify the singular points of the equation $x^3(x^2-1)y'' + 2x^4y' + 4y = 0$.

f) Compute the integral $\int_c xydx$ along the arc of the parabola $x=y^2$ from (1, –1) to (1, 1).

g) Obtain the differential equation of all conics whose axes coincide with the axes of coordinates.

h) Use the convolution theorem to evaluate

$$L^{-1}\left\{\frac{1}{(p+1)(p-1)}\right\}$$

i) Show that the repeated limits exist at the origin and are equal but the simultaneous limit does not exist for the following function

$$f(x,y)=\begin{cases}1, \text{ if } xy\neq 0\\ 0 \text{ if } xy=0\end{cases}$$

j) The function f is defined on $[0,\infty[$ by $f(x)=(-1)^{n-1}, n-1\leq x<n, n\in N$. Show that the integral $\int_0^\infty f(x)dx$ does not converge.

k) Compute $\int_{-1}^{1} f\,dx$, where $f(x)=|x|$.

l) If f is integrable on [a, b], then show that f^2 is also integrable on [a, b].

3. Answer any **five** questions: 6×5=30

a) Solve the equation

$$\frac{d^2y}{dx^2}+(x-1)^2\frac{dy}{dx}-4(x-1)y=0 \quad \text{in series}$$

about the point x=1.

b) i) Solve : $(y^2+z^2-x^2)p-2xyq+2zx=0$

ii) Show that the point of infinity is a regular singular point of the equation $x^2y''+(3x-1)y'+3y=0$. 4+2

c) i) Prove that the set C[a, b] of all real-valued functions continuous on the interval [a, b] with the function d defined by

$$d(f,g)=\left(\int_a^b f(x)-g(x)^2\,dx\right)^{\frac{1}{2}} \text{ is a metric}$$

space.

ii) Show that in any metric space (x, d) the intersection of a finite number of open sets is open. 4+2

d) i) If f_x and f_y are both differentiable at a point (a, b) of the domain of definition of a function f, then prove that

$$f_{xy}(a,b)=f_{yx}(a,b)$$

ii) Let $f(x,y)=\dfrac{xy(x^2-y^2)}{x^2+y^2}, (x,y)\neq(0,0)$

$f(0,0)=0.$

Then show that at the origin $f_{xy}\neq f_{yx}$.

4+2

e) Show that $\int_2^\infty \frac{\cos x}{\log x}dx$ is conditionally convergent.

f) Obtain the Fourier series in the interval $\left[-\frac{1}{2},\frac{1}{2}\right]$ of the function f given by

$$f(x)=\begin{cases} x-[x]-\frac{1}{2} & \text{when } x \text{ is not an integer} \\ 0, & \text{otherwise} \end{cases}$$

where [x] is the greatest integers $\le x$.

g) Prove that the function $f(x,y)=\sqrt{|xy|}$ is not differentiable at the point (0,0) but that f_x and f_y both exist at the origin and have the value 0. Hence deduce that these two partial derivatives are continuous except at the origin.

4+2

h) Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the conditions $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$ and $z=x+y$.

4. Answer any **three** questions: 15×3=45

a) i) Solve in series the differential equation $(2x+x^3)y''-y'-6xy=0$. For what values of x, the series so obtained are convergent?

ii) Evaluate:

$$\iint_E x^{m-1}y^{n-1}(1-x-y)^{p-1}\,dx\,dy, m\ge 1, n\ge 1, p\ge 1$$

where E is the region bounded by $x=0, y=0, x+y=1$.

iii) Prove that a series of functions Σf_n will converge uniformly and absolutely on [a,b] if there exists a convergent series ΣM_n of positive numbers such that for all $x\in[a,b]$, $|f_n(x)|\le M_n$ for all n.

(4+2)+6+3

b) i) Show that continuous image of a compact set is compact.

ii) Let (X, d) be a metric space. Then show that any disjoint pair of closed sets in X can be separated by disjoint open sets in X.

iii) Evaluate $\iint_E \sin\left(\frac{x-y}{x+y}\right)dxdy$, where E is the region bounded by the co-ordinate axes and x+y=1 in the first quadrant.

5+5+5

c) i) State and prove Taylor's theorem for functions of two variables with remainder after n terms.

ii) If $f(x,y)=\sqrt{|xy|}$ then prove that Taylor's expansion about the point (x,x) is not valid in any domain which includes the origin.

iii) Discuss the convergence of $\int_0^1 \log\sqrt{x}\,dx$ and hence evaluate it.

(1+5)+4+(2+3)

d) i) Prove that a bounded function f is integrable on [a, b] iff for every $\varepsilon > 0$ there exists a partition P of [a, b], such that $U(P,f) - L(P,f) < \varepsilon$.

ii) If f and g are integrable on [a,b] and g keeps the same sign over [a, b], then show that there exists a number μ lying between the bounds of f such that

$$\int_a^b fg\,dx = \mu\int_a^b g\,dx.$$

iii) Prove that lim I_n, where

$I_n = \int_0^\delta \frac{\sin nx}{x}dx, n \in N$ exists and equal to $\pi/2$.

6+3+6

e) i) The roots of the equation in λ

$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$

are u, v and w. Then prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -2\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}.$$

ii) If

$f(x,y,z) = (a^2x^2 + b^2y^2 + c^2z^2)/x^2y^2z^2$,

where $ax^2 + by^2 + cz^2 = 1$ and a, b c are positive, show that the minimum value of f(x,y,z) is given by

$$x^2 = \frac{u}{2a(u+a)}, y^2 = \frac{u}{2b(u+b)}, z^2 = \frac{u}{2c(u+c)}$$

where u is the positive root of the equation

$u^3 - (bc + ca + ab)u - 2abc = 0$.

iii) If V is a function of two variables x and y and $x = r\cos\theta$, $y = r\sin\theta$ then prove that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r}\frac{\partial V}{\partial r}$.

5+6+4

f) i) Solve:

$$z(x+y)\frac{\partial z}{\partial x} + z(x-y)\frac{\partial z}{\partial y} = x^2 + y^2.$$

ii) Use Laplace transforms to solve the following problem:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{-t},\ y(0) = y'(0) = 0.$$

iii) Solve the partial differential equation $px + qy = pq$ by Charpits method.

5+5+5
