U.G. 6th Semester Examination - 2021

MATHEMATICS

[HONOURS]

Discipline Specific Elective (DSE)

Course Code : MATH-H-DSE-T-03B

(Number Theory)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) State prime number theorem.

b) What is Goldbach conjecture?

c) Prove that $2^{4n}-1$ is divisible by 15.

d) State Fermat's little theorem.

e) Prove that $1835^{1910}+1986^{2061}\equiv 0(\text{mod } 7)$.

f) If p is a prime, prove that for any integer a, $p\mid a^{p}+(p-1)!a$.

g) Factor the number $2^{11}-1$ by Fermat's factorization method.

h) When n = 14, 206, show that $\sigma(n)=\sigma(n+1)$.

i) Verify that $\tau(n)=\tau(n+1)=\tau(n+2)$ holds for n = 3655.

j) Show that 1000! terminates in 249 zeros.

k) Determine the highest power of 3 dividing 80!.

l) Evaluate the Legendre symbol $\left(\frac{3658}{12703}\right)$.

m) Show that the prime divisors $p\neq 5$ of the integer n^2+n-1 are of the form $10k+1$ *or* $10k+9$.

n) Prove that 3 is a primitive root of all integers of the form 7^k and 2.7^k.

o) For a positive integer n, prove that $\sum_{d\mid n}\frac{\mu^2(d)}{\varphi(d)}=\frac{n}{\varphi(n)}$.

2. Answer any **four** questions: 5×4=20

a) Prove that if the congruence $x^2\equiv a(\text{mod } 2^n)$, where a is odd and $n\geq 3$, has a solution, then it has exactly four incongruent solutions.

b) Let p be an odd prime. Show that the equation $x^2 + py + a = 0$, $\gcd(a, p) = 1$ has an integral solution if and only if the Legendre symbol $\left(\frac{(-a)}{p}\right) = 1$.

c) Use the fact that each prime p has a primitive root and give a proof of Wilson's theorem.

d) If n is a square-free integer, prove that $\sum_{d|n} \sigma(d^{k-1})\varphi(d) = n^k$ for all integers $k \geq 2$.

e) Establish that for positive integers

m and n, $\varphi(m)\varphi(n) = \varphi(\gcd(m, n))\varphi(lcm(m, n))$

f) Show that if $\gcd(a, n) = \gcd(a-1, n) = 1$, then $1 + a + a^2 + \ldots + a^{\varphi(n)-1} \equiv 0 \pmod{n}$.

3. Answer any **two** questions: 10×2=20

a) Obtain three consecutive integers, the first of which is divisible by a square, the second by a cube, and third by a fourth power.

b) Let the positive integer n be written in terms of powers of the prime p so that we have $n = a_k p^k + \ldots + a_2 p^2 + a_1 p + a_0$, where $0 \leq a_i < p$.

Show that the exponent of the highest power of p appearing in the prime factorization of $n!$ is $\frac{n - (a_k + \ldots + a_2 + a_1 + a_0)}{p-1}$.

c) If the integer $n > 1$ has the prime factorization $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$, establish

$$\sum_{d|n} d\varphi(d) = \Pi_{i=1}^{r} \left(\frac{p_i^{2k_i+1} + 1}{p_i + 1}\right) \text{ and}$$

$$\sum_{d|n} \mu(d)\varphi(d) = \Pi_{i=1}^{r} (2 - p_i).$$

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