2021
MATHEMATICS
[HONOURS]
Paper : I

Full Marks : 100 Time : 4 Hours

The figures in the right-hand margin indicate marks.

Symbols, notations have their usual meanings.

1. Answer any **five** questions: 1×5=5

a) Determine the value of λ for which the vectors $-3\vec{i}+4\vec{j}+\lambda\vec{k}$ and $-6\vec{i}+\vec{j}-11\vec{k}$ are perpendicular.

b) Find the smallest positive integer n, for which $\left(\frac{1+i}{1-i}\right)^{2n}=1$.

c) If $A=\begin{pmatrix} 0 & \sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix}$, then show that A^2 is symmetric.

d) Find the Cartesian equation of the curve whose polar equation is given by $r^2=a^2\cos 2\theta$.

e) Given that $\omega^n+\omega^{2n}+1=0$ where ω is an imaginary cube root of unity and n is an integer. Is n divisible by 3? Give reason.

f) Is $f:\mathbb{R}\to\mathbb{R}$, given by f(x)=logx, $x\in\mathbb{R}$, a mapping where $\mathbb{R}$ is the set of all real numbers.

g) If α, β, γ are the roots of the equation $ax^3+bx^2+cx+d=0$, find the value of $\Sigma\alpha^2$, where $\alpha\neq 0$.

2. Answer any **ten** questions: 2×10=20

a) Without expanding find the value of the determinant

$$\begin{vmatrix} 7 & 12 & -3 \\ 9 & 14 & -1 \\ 8 & 13 & -2 \end{vmatrix}$$

b) Determine the eigenvalues of the matrix

$$\begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix}$$

c) Find the point of intersection of the straight line $\frac{x+1}{1}=\frac{y+3}{3}=\frac{z-2}{-2}$ and the plane $3x+4y+5z=5$.

d) For what value of λ, does the equation $xy+5x+\lambda y+15=0$ represent a pair of straight lines?

e) If $\left|\vec{\alpha}+\vec{\beta}\right|=\left|\vec{\alpha}-\vec{\beta}\right|$, then show that $\vec{\alpha}$ is perpendicular to $\vec{\beta}$.

f) For any two complex numbers z and z_0 show that $\arg(z-z_0)=-\arg(\overline{z-z_0})$.

g) Is $f:\mathbb{R}\to\mathbb{R}$, given by $f(x-1)=\log(x-1)$, $x\in\mathbb{R}$, a mapping where $\mathbb{R}$ is the set of all real numbers?

h) Find the equation whose roots are equal in magnitude but opposite in sign to the roots of

$$x^4+3x^3-7x^2+2x+1=0\cdot$$

i) If the binary operation $*$ be defined on I, the set of all integers by $a*b=a+b+1$, $a, b\in I$, find the identity element with respect to the operation $*$.

j) Verify whether the vectors $(2, 1, 0)$, $(1, 1, 0)$, $(4, 2, 0)$ of $\mathbb{R}^3$ are linearly dependent or independent.

k) Show whether the mapping $f:Z\to Z$ defined by $f(x)=|x|$, $x\in Z$ (=the set of all integers) is onto or not. Justify your answer.

l) Is it possible that all the roots of the equation $2x^3-11x^2+28x-24=0$ be complex? Justify your answer.

3. Answer any **five** questions: 6×5=30

a) i) Define Cyclic group. Show that every proper subgroup of a group of order 6 is cyclic.

ii) Show that the order of every subgroup of a finite group G is a divisor of the order of G. 3+3

b) i) Find the angle through which a set of rectangular axes must be turned without the change of origin so that xy term is removed from the equation $7x^2+4xy+3y^2=0$.

ii) Show that the lines $\frac{x+3}{4}=\frac{y-5}{-1}=\frac{z+7}{2}$, lies in the plane $x-2y-3z-8=0$. 3+3

c) i) If the two pair of lines $x^2-2pxy-y^2=0$ and $x^2-2qxy-y^2=0$ be such that each pair bisects the angle between the other pair, prove that $pq=-1$.

ii) Find the equation of the plane passing through the line $2x-y=0=3z-y$ and perpendicular to the plane $4x+5y-3z=0$. 3+3

d) i) Find all the roots of the equation $z^4-(1-z)^4=0$.

ii) Prove that $\sin\left(i\log\frac{x-iy}{x+iy}\right)=\frac{2xy}{x^2+y^2}$, x, y are real numbers. 3+3

e) i) Solve the cubic $x^3-12x+65=0$ by Cardan's method.

ii) If α, β, γ are the roots of the equation $x^3+x^2+x-2=0$, find the value of $\Sigma\frac{1}{\alpha}$, $\Sigma\frac{1}{\alpha\beta}$ and $\Sigma\frac{1}{\alpha^2}$. 3+3

f) i) Given that A and B are two matrices such that AB=A and BA=B. Show that A^T and B^T are both idempotent.

ii) Determine the values of α, β, γ when $\begin{pmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{pmatrix}$ is orthogonal. 3+3

g) i) Prove that

$$\begin{vmatrix} 1+a_1 & 1 & 1 \\ 1 & 1+a_2 & 1 \\ 1 & 1 & 1+a_3 \end{vmatrix} = a_1a_2a_3\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right).$$

ii) Solve the following system of equations by Cramer's rule:

$$2x-z=1$$
$$2x+4y-z=1$$
$$x-8y-3z=-2.$$

3+3

h) i) Show that a mapping $f:x\to y$ is invertible if and only if f is a bijection.

ii) Give an example of a mapping $f:Z\to Z$ which is injective but not surjective. Here Z denotes the set of all integers. 3+3

4. Answer any **three** questions: 15×3=45

a) i) Find the equation of a sphere which passes through the origin and makes equal intercepts of unit length on the axes.

ii) Find the equation of the right circular cylinder which passes through the point $(3,-1,1)$ and has the straight line $\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z-2}{1}$ as its axis.

iii) If the straight line $r\cos(\theta-\alpha)=p$ touches the parabola $\frac{l}{r}=1+\cos\theta$, show that $p=\frac{1}{2}\sec\alpha$. 5+5+5

b) i) Prove that a subgroup H of a group G is normal if and only if $aHa^{-1} = H$ for all $a \in G$.

ii) If the three vectors $\vec{\alpha} = a_1\vec{i} + b_1\vec{j} + c_1\vec{k}$, $\vec{\beta} = a_2\vec{i} + b_2\vec{j} + c_2\vec{k}$, $\vec{\gamma} = a_3\vec{i} + b_3\vec{j} + c_3\vec{k}$ are coplanar, show that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

iii) Find the equation of the tangent plane to the quadratic $3x^2 + 2y^2 - 6z^2 = 6$ which passes through the point $(3, 2, -3)$ and is parallel to the line $x = y = -z$.

5+5+5

c) i) Find the special roots of $x^9 - 1 = 0$. Deduce that $2\cos\frac{2\pi}{9}$, $2\cos\frac{4\pi}{9}$, $2\cos\frac{8\pi}{9}$ are the roots of the equation $x^3 - 3x + 1 = 0$.

ii) Solve the reciprocal equation $3x^6 + x^5 - 27x^4 + 27x^2 - x - 3 = 0$ by reducing to its standard form.

iii) If $\sin\alpha + \sin\beta + \sin\gamma = 0 = \cos\alpha + \cos\beta + \cos\gamma$, then prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2} = \sin^2\alpha + \sin^2\beta + \sin^2\gamma.$$

5+5+5

d) i) If $A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$, show that $A^3 - 6A - 9I_3 = 0$. Hence obtain a matrix B such that $BA = I_3$ where I_3 denotes the unit matrix of order 3.

ii) Find λ for which the system of equations

$$(4+\lambda)x + 2y + 2z = 0$$
$$2x + (4+\lambda)y + 2z = 0$$
$$2x + 2y + (4+\lambda)z = 0$$

has non-zero solutions.

iii) If A is a real skew symmetric matrix and I+A is non-singular, prove that $(I+A)^{-1}(I-A)$ is orthogonal. 5+5+5

e) i) Reduce the equation

$$7x^2 - 2xy + 7y^2 - 16x + 16y - 8 = 0$$

to canonical form and determine the nature of the conic.

ii) If $6x = 3y = 2z$, represents one of a set of three mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$, find the equation of the other two.

iii) A variable plane at a distance p from the origin meets the axes at A, B, C. Show that the locus of the centroid of the tetrahedron OABC is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{16}{p^2}.$$ 5+5+5
