2022
MATHEMATICS
[HONOURS]
Paper : VII

Full Marks : 100 Time : 4 Hours

The figures in the right-hand margin indicate marks.
Symbols and notations have their usual meanings.

GROUP–A

1. Answer any **five** questions: 1×5=5

a) Define sampling distribution of a statistics.

b) Write axioms of probability.

c) Define a mixed tensor of rank 2.

d) When a function f is called analytic at a point Z_0?

e) What is discrete distribution?

f) What do you mean by analytic function?

g) For a continuously differentiable scalar point function $\phi(x, y, z)$ write the geometrical interpretation of ϕ.

h) If the equation $a^i_j v_i = \alpha v_j$ holds for every covariant vector v_i where a is a scalar, show that $a^j_i = \alpha S^i_j$.

GROUP–B

2. Answer any **ten** questions: 2×10=20

a) If $P(A+B)=\frac{5}{6}$, $P(AB)=\frac{1}{3}$ and $P(\overline{B})=\frac{1}{2}$ then prove that A and B are independent.

b) Show that S^i_j is a mixed tensor of rank 2.

c) If $x=\frac{1}{2}\log\left(x^2+y^2\right)$ is harmonic then find its harmonic conjugate.

d) If $\vec{r}=a\cos t\,\hat{i}+a\sin t\hat{j}+bt\hat{k}$ find $\left|\frac{d^2\vec{r}}{dt^2}\right|$.

e) Show that $\nabla^2\left(\frac{1}{r}\right)=0$ where $r=\sqrt{x^2+y^2+z^2}$ and $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$.

f) Given an example with justification that two events are independent but not mutually exclusive.

g) State central limit theorem.

h) For a Random variable X with mean m and variance σ^2, state Tchebycheff's inequality.

i) Find the condition such that the bilinear transformation $\omega = \frac{az+b}{cz+d}$ has only one fixed point.

j) Show that $\text{curl}\{(\vec{a}.\vec{r})\vec{a}\} = \vec{0}$, where $\vec{a}$ is a constant vector.

k) Verify whether $f(z) = \log(z)$ is analytic at z=0.

l) Prove that the determinant of a tensor of type (1, 1) is an invariant.

GROUP–C

3. Answer any **five** questions: 6×5=30

a) Show that the function $f(z) = e^{z^4}\ (z \neq 0)$ and $f(0) = 0$ is not analytic at z = 0 although Cauchy-Riemann equations are satisfied at the point.

b) If the probability distribution of a discrete random variable X is given by $P(X = x) = ke^{-t}(1-e^{-t})^{x-1},\ x = 1, 2, \ldots \infty$, find the value of K and also the mean and variance of X.

c) If $b^{ij}u_iu_j$ is an invariant for arbitrary convariant vector u_i then show that $b^{ij} + b^{ii}$ is a contravariant tensor of rank 2.

d) State Green's theorem. Verify Green's theorem in plane $\oint_c (x^2 + y^3)dx + (x^3 + y^2)dy$ where C is the boundary of the pentagon with vertices (0, 0), (1, 0), (2, 1) (1, 2) and (0, 1).

e) Find Maximum Likelihood estimates of μ and σ^2 for $N(\mu, \sigma^2)$ population.

f) The probability density function of random variable X is given by

$$f(x) = Ae^{-(x^2+2x+3)},\ -\infty < x < \infty.$$

Calculate E(X) and Var(X), if exist.

g) Suppose a force field is given by $\vec{F} = (2x - y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$. Show that the work in moving a particle once around a circle C in the xy plane with its centre at the origin and radius 3 is 18π.

h) If $b^{ij}u_iu_j$ is an invariant for arbitrary covariant vector u_i then show that $b^{ij} + b^{ji}$ is a contravariant tensor of rank 2.

GROUP–D

4. Answer any **three** questions: 15×3=45

a) i) Applying central limit theorem to a sequence of random variables with Poisson distribution, prove that $\lim_{n \to \infty} e^{-n} \sum_{r=0}^{n} \frac{n^r}{r^!} = \frac{1}{2}$.

ii) The random variables X, Y are normally correlated with correlation coefficient ρ. Prove that $\frac{X}{\sigma x} + \frac{Y}{\sigma y}$ and $\frac{X}{\sigma x} - \frac{Y}{\sigma y}$ independent normal variates with variances $2(1+\rho)$ and $2(1-\rho)$ respectively.

iii) Assume that $A^i B_i$ is invariant for all contravariant vector A^i. Then show that B_i is covariant vector. 5+5+5=15

b) i) Show that a second order covarient tensor can be expressed as a sum of a symmetric and a skew-symmetric tensor.

ii) Verify Stoke's theorem for $\vec{F} = (2x+y)\hat{i} + yz^2\hat{j} + y^2z\hat{k}$ where S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and C is the boundary.

iii) Verify the divergence theorem for $\vec{A} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$. 5+5+5=15

c) i) Define confidence interval. Find the confidence interval for population mean m for normal (m, σ) population where σ is known.

ii) Two independent samples of sizes 8 and 7 contained the following values:

Sample 1 :	19	17	15	21	16	18	16	14
Sample 2 :	15	14	15	19	15	18	16	

Is the difference between the sample means significant? Test it at 5% level of significance

Given $t_{0.05}(\gamma = 13) = 2.16$.

iii) Find Maximum Likelyhood estimate of μ and σ^2 for $N(\mu, \sigma^2)$ population.

5+5+5=15

d) i) The mean value of a random sample of 60 items was found to be 145, with an SD of 40. Find the 95% confidence limits for the population mean. What size of sample is required to estimate the population mean within 5 of its actual value with 95% or more confidence, using sample mean.

Given $P(|Z| \le 1.96) = 0.95$.

ii) Write the definition of consistent and unbiased estimator of a parameter. Show that although sample variance is consistent estimator of population variance it is not unbiased estimator. Find the unbiased estimator of population variance.

iii) A random sample of size 10 is taken from a normal population and the following values where calculated for the variable (x) under study:

$$\sum_{i=1}^{10} x_i = 620, \sum_{i=1}^{10} x_i^2 = 39016$$

Test the null-hypothesis H_0: σ =8 against H_1: σ >8 on the basis of the above data. It is given that $\chi^2_{0.05} = 16.92$ with 9 degrees of freedom. 5+5+5

e) i) State and prove Cauchy-Riemann conditions of a function to be analytic in polar co-ordinate system.

ii) Find a bilinear transformation with map the unit circle $|z| \le 1$ onto the unit circle $|w| \le 1$.

iii) Find the analytic function of which the following function is the real part $\frac{\sin 2x}{\cosh 2y + \cos 2x}$. 5+5+5

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