U.G. 4th Semester Examination - 2022

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-9

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) If E be any subset of $\mathbf{R}^n$ and $f : E \to \mathbf{R}^n$ such that $f(x) = \|x\|^2$, find the directional derivative of f at the origin.

b) Find the points (x, y) and the directions for which the directional derivative of $f(x,y) = 3x^2 + y^2$ has its largest value, where (x, y) are restricted to be on the circle $x^2 + y^2 = 1$.

c) If $f(x,y) = \int_0^{\sqrt{xy}} e^{-t^2}\, dt$ for $x, y > 0$, compute $\frac{\partial f}{\partial x}$.

d) If $f(x,y) = \sqrt{|xy|}$, prove that $\frac{\partial f}{\partial x}(0,0) = 0$.

e) For the multiplicatively separable function $u(x,y) = f(x)g(y)$, prove that $u\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$.

f) Show that for the function $f(x,y) = 2(x-y)^2 - (x^4 + y^4)$, the point (0, 0) is not an extreme point.

g) Find the value of α for which the function

$$f(x,y) = \begin{cases} (x^2+y^2)\{\ln(x^2+y^2)+1\} & \text{if } (x,y) \neq (0,0) \\ \alpha & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at (0, 0).

h) If $F(x,y) = f\{x + g(y)\}$, show that $\frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial x^2}$, where f and g are functions of x and y respectively.

i) If $g(u,v) = f(u^2 - v^2)$, where $f : \mathbf{R} \to \mathbf{R}$ is twice differentiable, prove that $\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = 4(u^2+v^2)f''(u^2-v^2)$.

j) If $\vec{F} = (2x - ayz)\hat{i} + (2y - zx)\hat{j} + (2z - bxy)\hat{k}$ is irrotational, find the values of a and b.

k) Show that the force field $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is conservative.

l) Prove that the vector $\vec{F} = \frac{\vec{r}}{r^3}$ is solenoidal as well as irrotational, where $r = |\vec{r}|$.

m) Show that the work done by the force $\vec{F} = 4y\hat{i} - 3xy\hat{j} + z^2\hat{k}$ in moving a particle over the circular path $x^2 + y^2 = 1,\ z = 0$ from (1, 0, 0) to (0, 1, 0) is $-(\pi + 1)$.

n) Use Stoke's theorem to prove that $\int \vec{r}.d\vec{r} = 0$.

o) Evaluate $\int_0^\infty \int_0^\infty \frac{e^{-y}}{y} dx\ dy$ by changing the order of integration.

p) Evaluate $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z} dx\ dy\ dz$.

q) Prove that $\int_C [(2xy^3 + y)dx + (3x^2y^2 + 2x)dy] = \pi$ by using Green's theorem, where C represents the circle $x^2 + y^2 = 1$.

2. Answer any **four** questions: 5×4=20

a) Let $f : [a, b] \to \mathbf{R}$ and $g : [c, d] \to \mathbf{R}$ be continuous and $h : U \to \mathbf{R}$ be defined as $h(x, y) = \max\{f(x), g(y)\}$ for all $(x, y) \in U$, where $U = \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\}$. Prove that $h(x, y)$ is continuous in U.

b) Show that the function

$$f(x,y) = \begin{cases} x^2 + y^2 & \text{if } x \text{ and } y \text{ are rational} \\ 0 & \text{otherwise} \end{cases}$$

is differentiable at (0, 0).

c) If a function $f(x, y)$ of two variables x and y when expressed in terms of new variables u and v defined by $x = u + v$ and $y = uv$ becomes $g(u, v)$, then show that

$$\frac{\partial^2 g}{\partial u^2} - 2\frac{\partial^2 g}{\partial u \partial v} + \frac{\partial^2 g}{\partial u^2} = (x^2 - 4y)\frac{\partial^2 g}{\partial y^2} - 2\frac{\partial g}{\partial y}.$$

d) Using Lagrange's method prove that the volume of the largest rectangular parallelopiped that can be inscribed in an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{8abc}{3\sqrt{3}}$.

e) Evaluate $\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \left(x - \frac{y}{2}\right) dxdy$ using the transformation $x = u+v,\ y = 2v$.

f) Show that $\iint \frac{xy}{\sqrt{1+2x^2}} dS = \frac{-1}{6\sqrt{2}}$, where

$S = \{(x, y, x^2 + y) \in \mathbf{R}^3 : 0 \leq x \leq y, 0 \leq x + y \leq 1\}$.

3. Answer any **two** questions: 10×2=20

a) i) Let $f : D \to \mathbf{R}$, where D is an open subset of $\mathbf{R}^2$, be such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exists in a neighbourhood of $(x_0, y_0) \in D$ and are continuous at (x_0, y_0). Prove that f is differentiable at (x_0, y_0). 5

ii) Examine whether the following limit exists 5

$$\lim_{(x,y)\to(0,0)} \frac{|y|^{|x|}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+\frac{|y|}{|x|}}}.$$

b) i) Verify whether the double limit and the two repeated limits of the function $f(x, y)$ exist at (0, 0), where 5

$$f(x,y) = \begin{cases} \frac{x+y}{x-y} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

ii) State and prove Schwartz's theorem on the commutativity of mixed partial derivatives of a function of two variables. 5

c) i) Find the extrema of $u = xyz$ subject to the constraints $x + y + z = 5$ and $xy + yz + zx = 8$. 5

ii) Prove that the double integral of f on the rectangle $S = [0, 1] \times [0, 1]$ exists and equal to 0, where

$$f(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}.$$ 5

d) i) Verify the divergence theorem for $\vec{F} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ taken over the region in the first octant bounded by $y^2 + z^2 = 9$, $x = 2$. 5

ii) Verify Green's theorem in a plane for $\oint_C \{(xy + y^2)dx + x^2dy\}$, where C is the closed curve of the region bounded by $y = x^2$ and $y = x$. 5

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